Kepler's equation and some of its pearls

Francisco G. M. Orlando, C. Farina, Carlos A. D. Zarro, and P. Terra

Citation: American Journal of Physics **86**, 849 (2018); doi: 10.1119/1.5055760
View online: https://doi.org/10.1119/1.5055760
View Table of Contents: http://aapt.scitation.org/toc/ajp/86/11
Published by the American Association of Physics Teachers

Kepler's equation and some of its pearls

Francisco G. M. Orlando,[a)] C. Farina,[b)] and Carlos A. D. Zarro[c)]
Instituto de Física, Universidade Federal do Rio de Janeiro, Avenida Athos da Silveira Ramos, 149, Centro de Tecnologia, Bloco A, Cidade Universitária, CEP: 21941-972 Caixa Postal 68528, Rio de Janeiro-RJ, Brazil

P. Terra[d)]
Colégio Pedro II, Campus Centro, Avenida Marechal Floriano, 80, Centro, CEP: 20080-001, Rio de Janeiro-RJ, Brazil

(Received 19 January 2018; accepted 1 September 2018)

Kepler's equation, rarely discussed in undergraduate textbooks, was enunciated by Johannes Kepler in his *Astonomia Nova*, published in 1609, much before the advent of the integral and differential calculus. The search for its solutions challenged the minds of brilliant researchers like Newton, Lagrange, Cauchy, and Bessel, among others. In this work, we start with a standard derivation of Kepler's equation and emphasize how it gave rise to new mathematics, like approximation methods, Bessel functions, and complex analysis. Then we apply it in two non-trivial examples. In the first one, we compute the distance reached by a projectile launched from a point at the equator of the rotating Earth. This result could be used to prove the rotation of the Earth without the need of a Foucault pendulum. In the second example, we show how two astronauts moving around the Earth along the same circular orbit could exchange a sandwich. These two apparently innocent problems are quite involved because their solutions demand the calculations of the time of flight.
https://doi.org/10.1119/1.5055760

I. INTRODUCTION

Every motion of a particle under the action of solely a central force takes place in a fixed plane. This is a direct consequence of the central character of the force which produces no torque relative to the center of force, so that the angular momentum of the particle with respect to this point is a constant of motion. This fact, together with the spherical symmetry, strongly suggests that we use polar coordinates to handle any central force problems.

The constancy of the norm of the angular momentum $L = |\mathbf{L}|$ allows one to write $L = mr^2\dot{\varphi}$, with m being the particle's mass. Hence, appropriately using this simple relation we can eliminate the time dependence of any central force problem and, particularly, Kepler's problem. This is precisely what the great majority of undergraduate textbooks in classical mechanics do when discussing motions under a central force $\mathbf{F} = F(r)\hat{\mathbf{r}}$.[1–3] It can be proved that $u := 1/r$ satisfies the orbit equation given by

$$u''(\varphi) + u(\varphi) = -\frac{m}{L^2u^2}F(1/u). \tag{1}$$

The solutions to Eq. (1) give us all the possible trajectories of the particle. Particularly, in Kepler's problem, the orbit equation tells us that for $L \neq 0$ the trajectories are ellipses (circles as particular cases), parabolas and hyperbolas, so all limited orbits are closed.

Although it is very important to know what the allowed trajectories in Kepler's problem are, this is not sufficient to answer many questions as, for instance, what is the range of a projectile launched from the surface of the rotating Earth. In fact, any problem which involves the calculation of the time of flight cannot be solved by the orbit equation alone.

As we shall see, it is Kepler's equation that provides the temporal solution of Kepler's problem. This transcendental equation is largely used in astronomy. The magic of Kepler's equation—being so simple in form, yet so complicated—its rich history and its broad possibility of application motivated us to popularize it for undergraduate students as well as to exemplify how to use it in some interesting problems. We hope this work will be useful for both professors and students. The former may use it as complementary material for undergraduate or even graduate courses on classical mechanics, whereas the latter may be inspired to elaborate and solve many other interesting problems. The richness of Kepler's equation itself makes its study worthwhile.

This article is organized as follows. In Sec. II, we introduce Kepler's equation, for which we present a geometrical deduction. Section III is a historical survey of Kepler's equation emphasizing how it gave rise to new mathematics. In Secs. IV and V, we apply the equation to solve two interesting and quite non-trivial problems, both involving the calculation of the time of flight of a particle under the influence of the Earth's gravitational field. Section VI is left for conclusions and final remarks.

II. KEPLER'S EQUATION

Kepler's problem consists in finding the possible motions for a particle subject to an inverse-square-law net force $\mathbf{F}(r) = -(k/r^2)\,\hat{\mathbf{r}}$, where $k > 0$. From the orbit equation, for non-zero angular momentum with respect to the center of force, the possible Keplerian orbits are conic sections. In particular, all the limited orbits are ellipses, which can be written as[1–3]

$$r(\varphi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}, \tag{2}$$

where ε is the eccentricity of the orbit ($0 < \varepsilon < 1$) and a is its semi-major axis. The angle φ is called the *true anomaly*, defined as $\varphi = 0$ at the pericenter. We may also write the (constant) magnitude of the angular momentum of the particle L from its relation to the areal velocity

$$\frac{dA}{dt} = \frac{L}{2m}. \tag{3}$$

Then, for elliptic orbits ($0 < \varepsilon < 1$), with semi-axes a and $b = a\sqrt{1-\varepsilon^2}$, and denoting one complete period of motion by τ, we may write

$$L = \frac{2\pi}{\tau} m a^2 \sqrt{1-\varepsilon^2}. \tag{4}$$

So far, only the relation between φ and r (i.e., the trajectory) has been given. In other words, we have not solved Kepler's problem completely, as we do not have an expression for the coordinates φ and r as functions of time. This is not by chance. Indeed, obtaining the complete solution to Kepler's problem is a more complicated endeavor. In Subsections II A and II B, we aim to fill this gap by introducing the so-called temporal Kepler's equation, widely used by astronomers in the case of elliptical orbits.

A. The need for a new angular variable: Eccentric anomaly

A natural procedure for obtaining the temporal solution of Kepler's problem would be to depart from the angular momentum with respect to the center of force $L = mr^2\dot{\varphi}$. By integrating this equation and using Eqs. (2) and (4), we obtain

$$\omega t = (1-\varepsilon^2)^{3/2} \int_0^{\varphi} d\varphi' (1+\varepsilon\cos\varphi')^{-2}, \tag{5}$$

$$= (1-\varepsilon^2)^{3/2} \sum_{n=0}^{\infty} (-1)^n (n+1)\varepsilon^n \int_0^{\varphi} d\varphi' (\cos\varphi')^n, \tag{6}$$

where we defined $\omega = 2\pi/\tau$ and chose $\varphi = 0$ at $t = 0$. Recall that the previous equation is valid for elliptic orbits, for which $0 < \varepsilon < 1$, so that the power series converges uniformly.

Although the integral in Eq. (5) is solvable by means of expanding its right-hand side in powers of ε as written in Eq. (6), there is a more convenient method of expressing the solution to this problem, which Kepler presented in *Astronomia Nova* in 1609. It consists in using an angular variable other than φ for the position along the orbit, which holds a more concise relation to time: the *eccentric anomaly* ψ, which, as we shall see, is measured from the geometric center of the orbit, instead of from the center of force.

From this point, we could algebraically define ψ, providing an equation that relates it to φ, then integrate Eq. (5). This method is presented in advanced textbooks such as Landau and Lifshitz,[4] Goldstein,[5] and Poisson and Will.[6] However, we will follow a more geometrical approach which is closer to the one followed by Kepler himself.[7–9] Other interesting derivations of Kepler's equation can be found in a paper by Yoshida[10] and in Szebehely's book on celestial mechanics.[11]

B. Geometrical derivation of Kepler's equation

Figure 1 shows the elliptic trajectory of a planet around the sun. Let us consider the center of force fixed at point S, which is one of the foci of the ellipse, and the position of the planet at an arbitrary instant to be at point P. We set $t = 0$ when the planet is at the pericenter A.

In order to obtain the time-dependent solution, we first introduce the eccentric anomaly ψ: an angle whose vertex is at the center $\mathcal{O}$ of the ellipse. To do so, we draw an auxiliary segment $\overline{NQ}$ perpendicular to the major axis $\overline{AA'}$ through point P so that Q belongs to the circumference circumscribed to the elliptical trajectory. The eccentric anomaly is the angle defined as $\psi = \measuredangle A\mathcal{O}Q$. We also define the *mean anomaly* $M = \omega t$ which will be used later in Kepler's equation.

The areal velocity is constant and, considering a full period, equal to $\pi ab/\tau = \omega ab/2$. Then,

$$t = \frac{\text{Area}(ASP)}{\omega ab/2}, \tag{7}$$

where ASP is the "slice" of the ellipse swept out by segment $\overline{SP}$ from $t = 0$ to an arbitrary instant t. Similarly, consider now Area(ASQ) as the area swept out by segment $\overline{SQ}$ over the same time interval. Then (as shown in detail in Appendix A)

$$\text{Area}(ASP) = \frac{b}{a}\,\text{Area}(ASQ), \tag{8}$$

which may be argued simply by the fact that, since for all t, $NP/NQ = b/a$, the ellipse is related to the circle through stretching along the direction of the minor axis.

Substituting Eq. (8) into Eq. (7), we obtain

$$t = \frac{2}{\omega ab}\frac{b}{a}\,\text{Area}(ASQ). \tag{9}$$

Area(ASQ) can be obtained from more elementary areas. Notice from Fig. 1 that the circular sector $A\mathcal{O}Q$ is the union of triangle $\triangle S\mathcal{O}Q$ and the region ASQ so that

$$t = \frac{2}{\omega a^2}\left(\text{Area}(A\mathcal{O}Q) - \text{Area}(\triangle S\mathcal{O}Q)\right)$$
$$= \frac{2}{\omega a^2}\left(\frac{a^2\psi}{2} - \frac{a^2\varepsilon\sin\psi}{2}\right), \tag{10}$$

where we used the fact that $\mathcal{O}S = \varepsilon a$. Finally, the previous result may be cast into the form

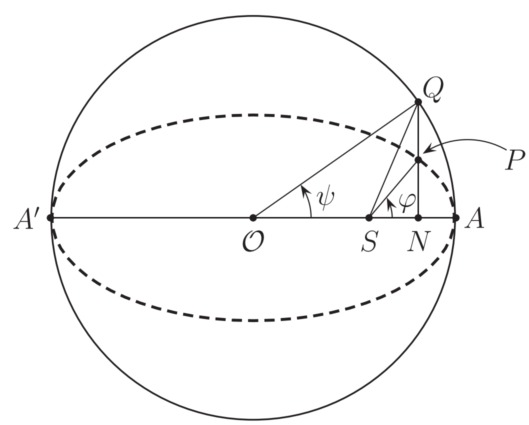

Fig. 1. The ellipse (dashed) with semi-axes $a = \mathcal{O}A$ and b is the orbit of a planet around a center of force S; the point closest to S in the trajectory (A) is the pericenter, whereas the farthest point (A') is the apocenter. A coplanar circle with radius a is drawn so that its center $\mathcal{O}$ is the same as the ellipse's center.

$$\omega t = \psi - \varepsilon \sin\psi, \tag{11}$$

which is Kepler's equation.

Despite its simple appearance, Eq. (11) does not allow for an analytical expression for ψ as a function of time, as it is transcendental. Thus it must be solved numerically by means of suitable approximation methods.

Let us relate angles ψ and φ. To do so, we recall from Fig. 1 that

$$\sin\psi = \frac{NQ}{a} = \frac{(a/b)\,NP}{a} = \frac{\sqrt{1-\varepsilon^2}\,\sin\varphi}{1+\varepsilon\cos\varphi}, \tag{12}$$

where we used Eqs. (2) and $NP/NQ = b/a$, with the relations $NP = r(\varphi)\,\sin\varphi$ and $b = a\sqrt{1-\varepsilon^2}$. Analogously, we have

$$\cos\psi = \frac{\mathcal{O}N}{a} = \frac{\mathcal{O}S + SN}{a} = \frac{\varepsilon + \cos\varphi}{1+\varepsilon\cos\varphi}, \tag{13}$$

where we additionally used the relations $\mathcal{O}S = \varepsilon a$ and $SN = r(\varphi)\cos\varphi$.

We conclude this section by expressing the position of the planet as a function of time. Keeping in mind that the temporal dependence of the eccentric anomaly is given by Kepler's equation (11), and exploiting the geometrical properties shown in Fig. 1, we have, in Cartesian coordinates with the origin placed at S

$$x(t) = ON(t) - OS = a[\cos\psi(t) - \varepsilon], \tag{14}$$

$$y(t) = NP(t) = \frac{b}{a} NQ(t) = a\sqrt{1-\varepsilon^2}\sin\psi(t), \tag{15}$$

or, in polar coordinates,

$$r(t) = a[1 - \varepsilon\cos\psi(t)], \tag{16}$$

$$\cos\varphi(t) = \frac{\cos\psi(t) - \varepsilon}{1-\varepsilon\cos\psi(t)}, \tag{17}$$

$$\sin\varphi(t) = \frac{\sqrt{1-\varepsilon^2}\,\sin\psi(t)}{1-\varepsilon\cos\psi(t)}, \tag{18}$$

where we used Eqs. (12) and (13).

III. BRIEF HISTORICAL SURVEY

Kepler's equation (11) was obtained for the first time in chapter 60 of Kepler's famous book *Astronomia Nova* published in 1609.[7,8] After presenting his first two laws, namely, that the orbit of a planet around the sun is an ellipse and the so-called area law, in the previous chapter, Kepler then derived his equation in a very geometric fashion fairly similar to the one presented in Sec. II B. Although in *Astronomia Nova* Eq. (11) is not written with mathematical symbols, Kepler was very precise in defining the mean anomaly and the eccentric anomaly.

Solving Kepler's equation (11) for the eccentric anomaly and then obtaining the true anomaly with the aid of the equations written in Eqs. (17) and (18) as a function of the mean anomaly, $M = \omega t$, is far from being a simple task. Indeed Kepler's equation (11) is a transcendental equation. These facts were noted by Johannes Kepler himself as is evident from his own words:[8]

> *It is enough for me to believe that I could not solve this a priori, owing to the heterogeneity of the arc and the sine. Anyone who shows me my error and points the way will be for me the great Apollonius.*

Hence, one must find some approximations to solve that equation. In spite of its apparent simplicity, this equation has challenged for centuries some notable mathematicians, such as Isaac Newton, Joseph Lagrange, Augustin-Louis Cauchy and Friedrich Bessel, among others. In their efforts to find a solution to Kepler's equation, some important breakthroughs in Mathematics occurred. A very nice and comprehensive historical survey on this subject has been made by Colwell.[12]

The first approximate solution to Kepler's equation was done by Kepler himself by using an iterative numerical method, which presumably converges to the correct solution. Unfortunately, as shown by Euler, this method is convenient—i.e., converges to the correct solution after few steps—only for very small eccentricities. An improvement to solve Kepler's equation numerically was done by Newton, published in *Principia* and it is now dubbed the Newton-Raphson method.[6] One can show that this method requires only a few iterations to get excellent numerical precision.

Another way to find solutions to Kepler's equation consists in making a power series solution of the mean anomaly M or the eccentricity ε. Lagrange showed that the power series solution of Kepler's equation converges only for small values of ε. Cauchy was not satisfied by this proof and introduced complex variables to deal with this problem. For the first time, a residue calculation and a systematic way of obtaining the convergence radius of a power series were presented. Rather than seek solutions in a power series of eccentricity, one can also propose putative solutions in Fourier series for the mean anomaly M. In 1818, this issue was tackled by Bessel, in a letter written to Olbers,[13] where he presented for the first time the functions henceforth known as Bessel functions.

Interestingly, there are some non-analytic solutions to Kepler's equation.[12] All are motivated by the need to find a fast method to determine the position of the planet at an arbitrary instant of time. These non-analytic approaches seek an approximate solution to Kepler's equation (11); some of them consider that the orbits do not satisfy Kepler's second law, others are neat geometrical constructions that rely on some approximation. One of the most interesting solutions was presented by Sir Christopher Wren, the architect of Saint Paul's Cathedral in London, who used the properties of the cycloid to find a solution of Kepler's equation.

Many other interesting methods to solve Kepler's equation have been developed since its appearance in 1609. Moreover, this issue continues to be a topic of research.[14–18] In order to apply Kepler's equation in concrete situations, we present in Secs. IV and V two interesting problems whose solutions would be extremely involved without it.

IV. PROJECTILE LAUNCHING FROM EARTH

The main purpose of this section is to illustrate Kepler's equation in the computation of the range achieved by a projectile launched from the surface of the Earth considering its rotation about its axis, that is, assuming a spinning Earth. For simplicity, we shall consider a projectile launched from the Earth's equator with an initial velocity belonging to the equatorial plane. As we shall see, the solution to this problem demands the knowledge of the projectile's time of flight, to be determined with the aid of Kepler's equation. This is an interesting problem for an undergraduate student for many reasons, one of which is the following: as we shall see, a projectile's range depends on the angular velocity of the Earth, so that, in principle, measurements of the projectile's range could be used to prove Earth's rotational movement, without the need of a Foucault pendulum (though a Foucault pendulum is much more convenient for this task).

Consider a projectile of mass m launched from a point at the equator of the Earth (supposed a homogeneous sphere of radius R and mass $M_\oplus \gg m$ centered at $\mathcal{O}$) with velocity $\mathbf{V}_0$ at an angle θ_0 with the planet's surface. The projectile's energy E is insufficient for it to escape the gravitational field—i.e., $0 < V_0 < V_e$, where $V_0 = |\mathbf{V}_0|$ is the magnitude of the projectile's initial velocity and $V_e = \sqrt{2GM_\oplus/R} \approx 11.2\,\text{km/s}$ is the escape velocity from the Earth's surface. If the particle's angular momentum relative to $\mathcal{O}$ is such that $|\mathbf{L}| \neq 0$, the projectile's trajectory will be an ellipse with one focus located at $\mathcal{O}$. The semi-major axis and eccentricity of the ellipse are determined by the knowledge of the mechanical energy E of the system and the projectile's angular momentum $\mathbf{L}$[2]

$$a = -\frac{GM_\oplus m}{2E}, \tag{19}$$

and

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{G^2 M_\oplus^2 m^3}}. \tag{20}$$

Let us then calculate the projectile's range s along the surface of the Earth. If the Earth did not spin it would suffice to find the intersections between the projectile's elliptical trajectory and the surface of the Earth, and then calculate the great-circle distance $s = R\delta$ between them, where δ is defined in Fig. 2, where we took advantage of the ellipse's symmetry to bisect δ. The polar axis $\mathcal{O}x$ was chosen as the major axis of the ellipse.

However, since the Earth *does* spin, to $s = R\delta$ we must add or subtract the distance the ground moves during the particle's flight, depending on the particle's initial velocity.

In what follows we will explicitly calculate the range for each case. For the sake of clarity, and to establish some basic concepts, we begin with the case of a stationary Earth.

A. Projectile launching from a stationary Earth

From Fig. 2, it is evident that $\delta/2 + \varphi_0 = \pi$, so that

$$\cos(\delta/2) = -\cos\varphi_0. \tag{21}$$

In order to relate δ to the semi-major axis a and the eccentricity ε, we set $r(\varphi_0) = R$ in Eq. (2) and then use Eq. (21)

$$R = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi_0} \Rightarrow \cos(\delta/2) = \frac{R - a(1-\varepsilon^2)}{\varepsilon R}. \tag{22}$$

In the last equation, $\cos(\delta/2)$ is written in terms of the geometrical parameters a and ε of the trajectory, but it is convenient to write this result in terms of V_0 and θ_0. With this purpose, we write the expressions of the magnitude of the particle's angular momentum with respect to the center of

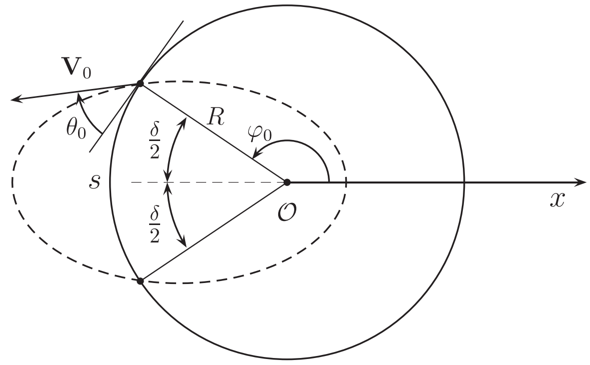

Fig. 2. The elliptical trajectory (dashed) of a projectile launched from the equator of the Earth (solid) with initial velocity $\mathbf{V}_0$. The true anomaly at the launching point φ_0 and the projectile's range s are also indicated.

the Earth and the mechanical energy of the system in terms of these quantities

$$L = mV_0 R\cos\theta_0 \quad \text{and} \quad E = \frac{mV_0^2}{2} - \frac{GM_\oplus m}{R}. \tag{23}$$

Defining the dimensionless constant ξ by

$$\xi = \frac{V_0^2 R}{GM_\oplus} = \frac{2V_0^2}{V_e^2}, \tag{24}$$

the substitution of Eq. (23) into Eqs. (19) and (20) leads to

$$a = \frac{R}{2-\xi} \tag{25}$$

and

$$\varepsilon = \sqrt{1 + (\xi^2 - 2\xi)\cos^2\theta_0}. \tag{26}$$

Substituting then Eqs. (25) and (26) into Eq. (22), we obtain

$$\cos(\delta/2) = \frac{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0/(2-\xi)}{\sqrt{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}} = \frac{1 - \xi\cos^2\theta_0}{\sqrt{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}}. \tag{27}$$

Finally, rearranging terms

$$s(\xi,\theta_0) = R\delta(\xi,\theta_0) = 2R\arccos\left(\frac{1 - \xi\cos^2\theta_0}{\sqrt{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}}\right). \tag{28}$$

This gives the exact projectile's range on a stationary Earth as a function of $\xi = 2V_0^2/V_e^2$ and θ_0. This is already quite interesting since, among other things, it shows that, for a fixed V_0 (a fixed ξ), the maximum range achieved by the projectile depends on θ_0. In Fig. 3, we plot the projectile's range normalized by its maximum value, s/s_{max}, as a function of θ_0 for different (fixed) values of ξ. It is evident from this figure that the projectile's range is maximum ($s/s_{max} = 1$) at different values of θ_0. Besides, complementary launching angles do not lead to the same projectile's range, as in the usual case for $\xi \ll 1$.

In fact, it is not difficult to obtain the launching angle that leads to the maximum projectile's range, which we denote by θ_{0m}. It suffices to maximize Eq. (28). As arccos is a monotonic function, we just need to extremize its argument. Doing that yields

$$\cos\theta_{0m} = \sqrt{\frac{1}{2-\xi}}. \tag{29}$$

A few comments are in order here. First note that in the limit $\xi \to 0$, $\cos\theta_{0m} \to \sqrt{1/2}$ so that $\theta_{0m} \to \pi/4$, as expected. Besides, as ξ increases, $\cos\theta_{0m}$ also increases and hence θ_{0m} decreases, as it is evident in Fig. 3. However, although $0 < V_0 < V_e$ implies $0 < \xi < 2$, the previous equation has solutions only for $0 < \xi \le 1$, since $0 \le \cos^2\theta_{0m} \le 1$. For $1 < \xi < 2$, Eq. (29) does not admit any solution, which means that for $1 < \xi < 2$, the projectile's range is a monotonic function of θ_0.

This can be interpreted nicely: $\xi = 1$ means that $V_0 = V_e/\sqrt{2}$, which the reader may verify is the velocity of a particle in a circular orbit with the radius of the Earth (a low-pass circular orbit). Hence, if a projectile is launched with $\xi = 1$, as the launching angle θ_0 diminishes, approaching zero, the projectile's range increases, approaching $2\pi R$. For $1 < \xi < 2$, the maximum projectile range will also occur at $\theta_0 = 0$ and will always be equal to $2\pi R$, since for these values of the projectile's launching velocity the orbits will always be ellipses with the perigee at the launching point (the only intersection point between the projectile's orbit and the Earth).

We finish this subsection by making a last self-consistency check on Eq. (28). We shall recover from this equation the projectile's range for very small initial velocities, for which the projectile will experience a constant gravitational field. It suffices to consider $\xi \ll 1$ in Eq. (27), in which we expand both sides of the previous equation in powers of δ (left hand side) and ξ (right hand side), and retain terms only up to second order. It is straightforward to show that

$$\begin{aligned} 1-\frac{1}{2!}\left(\frac{\delta}{2}\right)^2+\mathcal{O}(\delta^4) &= \left(1-\xi\cos^2\theta_0\right)\Big(1+\xi\cos^2\theta_0 \\ &\qquad -\frac{1}{2}\xi^2\cos^2\theta_0+\frac{3}{2}\xi^2\cos^4\theta_0+\mathcal{O}(\xi^3)\Big) \\ &\approx 1-\frac{\xi^2}{2}\cos^2\theta_0\left(1-\cos^2\theta_0\right) \\ &= 1-\frac{\xi^2}{8}\sin^2(2\theta_0), \end{aligned} \tag{30}$$

so that we immediately identify $\delta = \xi\sin(2\theta_0)$, which can be substituted in $s = R\delta$ to give the well known result for the range of projectiles moving near the surface of the Earth, namely, $s = \frac{V_0^2}{g}\sin(2\theta_0)$, where $g = \frac{GM_\oplus}{R^2}$ is the magnitude of the gravitational field at the surface.

B. Projectile launching from a rotating Earth

A more involved problem is that of determining the projectile's range considering a rotating Earth. It is clear, now, that we need to calculate the distance covered by the ground during the projectile's flight. For simplicity, we assume the Earth rotates with a constant angular velocity $\boldsymbol{\Omega}$ perpendicular to the equatorial plane and, thus, also to the launching velocity. Let $\mathbf{v}_0$ be the projectile's launching velocity with respect to a frame rotating rigidly with the Earth and let ζ be the smallest angle between $\mathbf{v}_0$ and the surface of the planet. Since, for our purposes, it is desirable to write the projectile's range in terms of $v_0 = |\mathbf{v}_0|$ and ζ, the first thing we need to do is to express V_0 and θ_0 in terms of v_0 and ζ.

Due to the rotation of the Earth, and denoting by $\mathbf{V}_t$ the velocity of the launching point at the equator, the Galilean composition of velocities allows us to write immediately $\mathbf{V}_0 = \mathbf{v}_0 + \mathbf{V}_t$, where $\mathbf{V}_t = \Omega R\hat{\mathbf{t}}$, with $\Omega = |\boldsymbol{\Omega}|$ and $\hat{\mathbf{t}} = \mathbf{V}_t/|\mathbf{V}_t|$. Figure 4 shows all these velocities as well as the relevant angles.

The algebraic relation between the angles and magnitudes of the velocities may then be written

$$V_{0\pm}^2 = (\mathbf{v}_0 + \mathbf{V}_t)^2 = v_0^2 + \Omega^2R^2 \pm 2v_0\Omega R\cos\zeta, \tag{31}$$

where the signs depend on whether the projectile is launched in the same (+) or opposite (–) direction of the rotation of the Earth. Notice that Fig. 4 was drawn for the first case, to avoid overloading it. From the same figure,

$$\sin\theta_{0\pm} = \frac{v_0}{V_{0\pm}}\sin\zeta. \tag{32}$$

Note that for $v_0\cos\zeta < \Omega R$, even for the case where the projectile is launched against the rotation, its motion will be towards the rotation.

Let us then proceed to calculate the projectile's range. Assuming, initially, that the projectile is launched in such a way that it falls ahead of the launching point, the projectile's range is given by

$$A = s - \Omega R t_{\mathrm{flight}}, \tag{33}$$

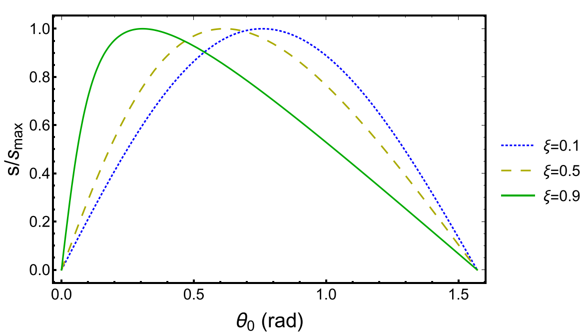

Fig. 3. (color online) Normalized range for different values of ξ as a function of the launching angle θ_0.

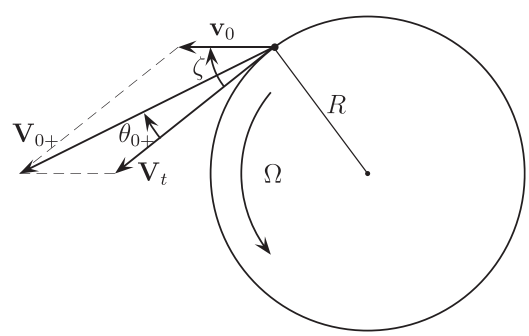

Fig. 4. Projectile launched in the direction of Earth's rotation, from a point on the equator of the Earth.

where $\Omega R t_{\text{flight}}$ is the distance covered by the launching point during the projectile's flight, to be determined with the aid of Kepler's equation (11), and s is given by Eq. (28).

Let t_0 be the instant the particle is launched and t_1, the instant it lands. Also, let ψ_0 and ψ_1 be the eccentric anomalies at the launching and landing points, respectively. The symmetry displayed in Fig. 5 makes it evident that $\psi_1 = 2\pi - \psi_0$.

Using the above results, *as well as Kepler's equation*, the projectile's time of flight is given by

$$\begin{aligned} t_{\text{flight}} &= t_1 - t_0 \\ &= \frac{1}{\omega}\{(\psi_1 - \varepsilon\sin\psi_1) - (\psi_0 - \varepsilon\sin\psi_0)\} \\ &= \frac{1}{\omega}\{2\pi - \psi_0 - \varepsilon\sin(2\pi-\psi_0) - \psi_0 + \varepsilon\sin\psi_0\} \\ &= \frac{2}{\omega}\{\pi - \psi_0 + \varepsilon\sin\psi_0\}, \end{aligned} \tag{34}$$

where ε has already been expressed in terms of V_0 and θ_0 through Eq. (26), with ξ defined by Eq. (24). Then, if we express ω and ψ_0 in terms of V_0 and θ_0 (which have already been written in terms of v_0 and ζ) we will have succeeded in expressing t_{flight} in terms of v_0 and ζ.

Kepler's third law applied to the projectile-Earth system states that $\frac{\tau^2}{a^3} = \frac{4\pi^2}{GM_\oplus}$, where τ would be the period of the projectile in its entire elliptical orbit if all the mass of the Earth were concentrated in its center. This equation can be written as $\omega^2 = \left(\frac{2\pi}{\tau}\right)^2 = \frac{GM_\oplus}{a^3}$, which, by substitution of Eq. (25), takes the form

$$\omega = 2\sqrt{\frac{2GM_\oplus}{R^3}}\left(1 - \frac{\xi}{2}\right)^{\frac{3}{2}}, \tag{35}$$

with ξ being given in Eq. (24).

It is convenient to express ψ_0 in terms of $\xi = V_0^2 R/(GM_\oplus)$ and θ_0. This lengthy but straightforward calculation (which is shown in detail in Appendix B) yields

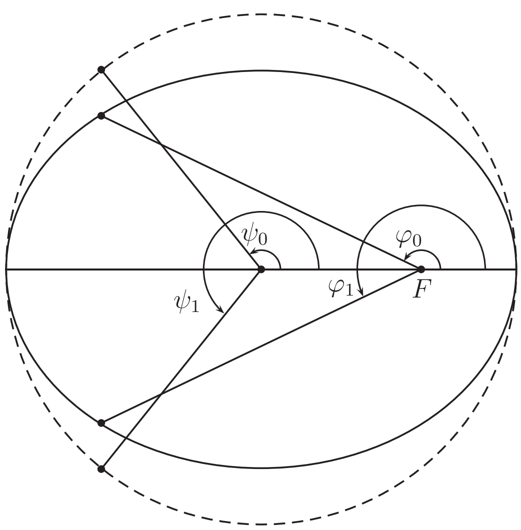

Fig. 5. True anomalies φ_0 and φ_1 at launching and landing points, respectively, and the corresponding eccentric anomalies, ψ_0 and ψ_1. To avoid overloading the figure, we omitted the Earth, whose center coincides with the focus F of the ellipse.

$$\sin\psi_0 = \sqrt{\frac{2\xi - \xi^2}{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}}\ \sin\theta_0, \tag{36}$$

$$\cos\psi_0 = \frac{\xi - 1}{\sqrt{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}}. \tag{37}$$

Since $\sin\alpha = \sin(\pi - \alpha)$, Eq. (36) allows for two solutions for ψ_0. From Eq. (37) it is evident that if $\xi \le 1$, then $\psi_0 \ge \pi/2$ and if $\xi > 1$, then $\psi_0 < \pi/2$, so that

$$\psi_0 = \begin{cases} \pi - \arcsin\left(\sqrt{\dfrac{2\xi-\xi^2}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}}\sin\theta_0\right) & \text{if } \xi \le 1 \\ \arcsin\left(\sqrt{\dfrac{2\xi-\xi^2}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}}\sin\theta_0\right) & \text{if } \xi > 1. \end{cases} \tag{38}$$

Then, the projectile's time of flight in Eq. (34) can be rewritten using Eqs. (35) and (36) as

$$t_{\text{flight}} = \sqrt{\frac{R^3}{2GM_\oplus}}\left(1 - \frac{\xi}{2}\right)^{-\frac{3}{2}}\left(\pi - \psi_0 + \sqrt{2\xi - \xi^2}\sin\theta_0\right). \tag{39}$$

In order to decide which expression for ψ_0 in Eq. (38) should be substituted in the previous equation, we must determine whether $\xi > 1$ or $\xi \le 1$.

Henceforth, we shall assume the condition $v_0 < \Omega R$. In this case, as $V_0 = |\mathbf{v}_0 + \Omega R\hat{\mathbf{t}}| < 2\Omega R$ and, using the numerical values $\Omega = 2\pi\,\text{rad/day}$, $R \approx 6400\,\text{km}$ and $V_e \approx 11.2\,\text{km/s}$, we have that $\xi = 2V_0^2/V_e^2 < 1$, so that the final expression for the projectile's time of flight is

$$\begin{aligned} t_{\text{flight}} = {} & \sqrt{\frac{R^3}{2GM_\oplus}}\left(1 - \frac{\xi}{2}\right)^{-\frac{3}{2}} \\ & \times \left(\arcsin\left(\sqrt{\frac{2\xi - \xi^2}{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}}\sin\theta_0\right)\right. \\ & \left. + \sqrt{2\xi - \xi^2}\sin\theta_0\right), \end{aligned} \tag{40}$$

regardless of whether the projectile is launched against or with the Earth's rotation.

Equation (40) is one of the main results of this section, to be used in the computation of the projectile's range. It is an exact result and recalling the definition $\xi = 2V_0^2/V_e^2$ along with Eqs. (31) and (32), t_{flight} can be explicitly expressed in terms of the norm of the projectile's velocity relative to the rotating Earth, v_0, and the angle ζ.

Before we proceed, let us check once again the self-consistency of our calculations by recovering the time of flight for projectiles launched with very small velocities, so that they experience a constant gravitational field and describe a parabolic trajectory. In this particular regime, we have $\xi \ll 1$, so that $\varepsilon\sin\psi_0 \approx \sqrt{2\xi}\sin\theta_0$. Also, since, near the Earth, the projectile describes a parabolic trajectory, we may write

$\varepsilon \approx 1$, so that $\sin\psi_0 \approx \sqrt{2\xi}\sin\theta_0$. Since, for small values of ξ, ψ_0 is close to π, $\sin\psi_0 = \sin(\pi - \psi_0) \approx \pi - \psi_0$. Therefore, $\psi_0 \approx \pi - \sqrt{2\xi}\sin\theta_0$. For small ξ, we have $\omega \approx 2\sqrt{2GM_\oplus/R^3}$. Substituting all these results into Eq. (34), we obtain $t_{\mathrm{flight}} \approx 2V_0 \sin\theta_0 R^2/(GM_\oplus) = 2V_0\sin\theta_0/g$, the well-known result for a constant gravitational field.

Although A depends on Ω, we could naively think that two projectiles launched with initial velocities relative to the Earth of the same magnitude v_0 and the same launching angle ζ, but in opposite directions, would have the same range, as it occurs in the usual case of small velocities. However, this is not true in the general case.

First of all, the condition that the projectile is launched towards the rotation of the Earth is not sufficient to determine if it will fall ahead (i.e., to the east) of the launching point. Notice that, from conservation of angular momentum, $\dot{\varphi} = \Omega R/r$. During the flight, $r > R \Rightarrow \dot{\varphi} < \Omega$, so the projectile will travel a smaller angular displacement than the surface. Then an upward throw ($\zeta = \pi/2$) would result in the projectile falling to the west of its launching point,[20] which means that for every v_0 there must be a critical angle $\zeta_c(v_0) < \pi/2$ (but, in Earth's case, very close to $\pi/2$) for which the projectile lands back in it.

Let A_+ be the projectile's range with launching velocity relative to the Earth $\mathbf{v}_0$ pointing east with an angle $\zeta < \zeta_c$ similarly to Fig. 4, so that it lands to the east of the launching point, and A_- the range of a similar projectile that differs from the first one only in that it points west (in this case $\mathbf{v}_0$ and $\mathbf{V}_t$ form an obtuse angle $\pi - \zeta$). Then A_+ is precisely given in Eq. (33), but calculated with $\xi_+ = 2V_{0+}^2/V_e^2$ and θ_{0+}, that is, $A_+ = s(\xi_+, \theta_{0+}) - \Omega R t_{\mathrm{flight}}(\xi_+, \theta_{0+})$. Figure 6 shows A_+ as a function of v_0 for $\zeta = \pi/4$. In this figure, we marked three reference values of v_0, namely, the velocity of a point at the Earth's equator, ΩR, the value of v_0 whose (local) horizontal component equals ΩR, and the launching velocity of a bullet for a typical firearm, approximately 1000 m/s.

The computation of A_- is more subtle for the following reasons. First, it must be computed with variables $\xi_- = 2V_{0-}^2/V_e^2$ and θ_{0-}. Second, for $0 < v_0 < \Omega R/\cos\zeta$, it is given by

$$A_- = \Omega R t_{\mathrm{flight}}(\xi_-, \theta_{0-}) - s(\xi_-, \theta_{0-}), \tag{41}$$

since in this case, although it falls to the west of the launching point, the projectile in effect still moves east. (Notice that this is also the case when pointing $\mathbf{v}_0$ eastwards, but with $\zeta > \zeta_c$.) On the other hand, for the case where $v_0 > \Omega R/\cos\zeta$, but such that V_{0-} is still smaller than V_e, A_- must be computed by

$$A_- = \Omega R t_{\mathrm{flight}}(\xi_-, \theta_{0-}) + s(\xi_-, \theta_{0-}), \tag{42}$$

since in this case the projectile moves in the direction opposite to the rotation of the Earth. We are now able to compare A_+ and A_- for two projectiles launched in opposite directions but with the same values of v_0 and ζ (note that ξ_+ and ξ_-, as well as θ_{0+} and θ_{0-} are not the same, as is evident from Eqs. (31) and (32). Figure 7 shows the difference $A_+ - A_-$ for two projectiles launched in opposite directions with the same launching angle $\zeta = \pi/4$ and the same launching velocities relative to the Earth v_0, as a function of v_0.

As anticipated, $A_+ \neq A_-$, contrary to what we might naively expect. If we use as projectiles to compute $A_+ - A_-$ the bullets shot from a typical weapon, like, for instance, an AR-15, and use a launching angle $\zeta = \pi/4$ for both shots in opposite directions, the difference in the corresponding ranges would be approximately 1.3 km if the shots occur at zero latitude, as shown in Fig. 7. In this sense, the rotation of the Earth could be proved without appealing to any Foucault pendulum, by only comparing projectile's ranges in opposite directions. Of course, for practical reasons, a Foucault pendulum is much more convenient for this task. The importance of the previous calculations lies, obviously, not in proving Earth's rotation, but in its relevant applications in the study of the motions of projectiles and satellites around the Earth, and in the Kepler problem in general.

V. HOW DOES AN ASTRONAUT SEND A SANDWICH TO ANOTHER ASTRONAUT?

As another application of Kepler's equation, let us solve a quite academic but very nice problem involving two astronauts, 1 and 2, moving along the same circular orbit of radius R_0 around the Earth. Neglect the gravitational attraction between the astronauts, so that both of them are under the influence of only the Earth's gravitational field. Suppose astronaut 1, who is ahead in the orbit of astronaut 2 by a given (but arbitrary) angular displacement ϕ, decides to send him a sandwich. How must he do it? A trivial solution would

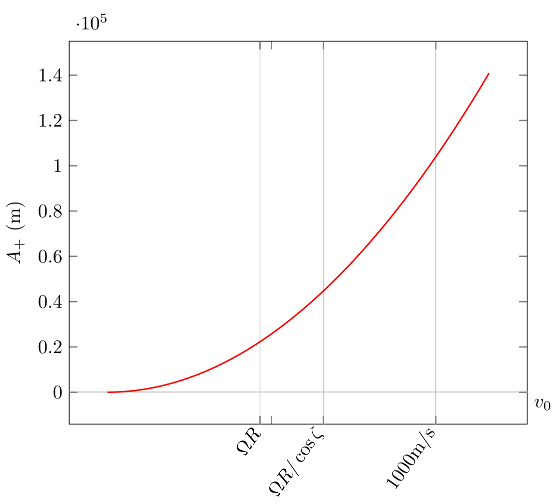

Fig. 6. Plot of A_+ as a function of v_0 for a fixed launching angle $\zeta = \pi/4$.

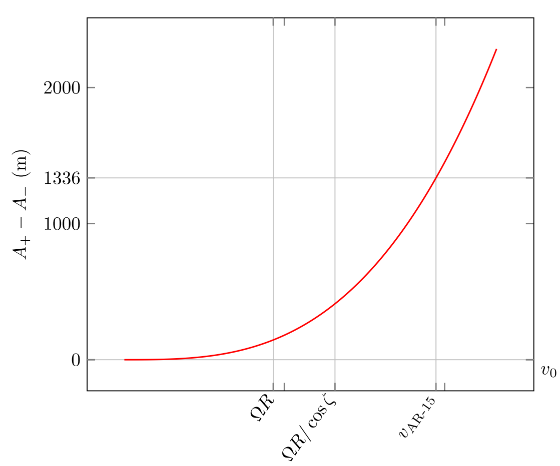

Fig. 7. Plot of $A_+ - A_-$ as a function of v_0 for equal launching angles $\zeta = \pi/4$, where we used $v_{\mathrm{AR-15}} = 973$ m/s as a reference for a launching velocity from a firearm.

be to send the sandwich along the same circular orbit as the astronauts, but in the opposite direction, as discussed by Walter Lewin.[19] However, this would not require knowledge of the time of flight.

In order to simplify things slightly, let us suppose that astronaut 1 throws the sandwich impinging upon it an outward radial impulse of magnitude mv_{r0}, where m is the mass of the sandwich, so that its launching velocity relative to an inertial reference frame is given by $\mathbf{V}_0 = V_A\hat{\theta} + v_{r0}\hat{\mathbf{r}}$, where $V_A = \sqrt{GM_\oplus/R_0}$ is the magnitude of the velocity of each astronaut in the circular orbit, as indicated in Fig. 8. In this figure, the positions of the astronauts are indicated at two instants: at the initial instant when the sandwich is thrown (positions 1 and 2) and when astronaut 2 catches it ($1'$ and $2'$).

We neglected the radial impulse received by astronaut 1 (as a consequence of Newton's third law) since its mass is much greater than m. It is worth emphasizing that this effect does not affect our result, since it does not alter the fact that astronaut 2 describes a circular orbit until the moment he catches the sandwich.

Let us denote by δ the angular displacement of the sandwich since it is thrown by the first astronaut (at position 1) until it is caught by the second astronaut (at position $2'$) and by t_{flight} the time interval for this process. As we shall see in a moment, it can be shown that $\delta = \pi$, regardless of the value of v_{r0} (note that we already used this information in Fig. 8).

Hence, for astronaut 2 to catch the sandwich in the least possible time interval, his angular displacement during the time interval t_{flight} must satisfy the relation

$$\pi + \phi = \Omega t_{\text{flight}}, \tag{43}$$

where $\Omega = V_A/R_0 = \sqrt{GM_\oplus/R_0^3}$ is the angular velocity of the astronauts in their circular orbits.

From the relation $\mathbf{V}_0 = V_A\hat{\theta} + v_{r0}\hat{\mathbf{r}}$ we immediately have

$$V_0^2 = V_A^2 + v_{r0}^2 = \frac{GM_\oplus}{R_0} + v_{r0}^2 = \frac{GM_\oplus}{R_0}\left(1 + \frac{v_{r0}^2 R_0}{GM_\oplus}\right), \tag{44}$$

and, consequently,

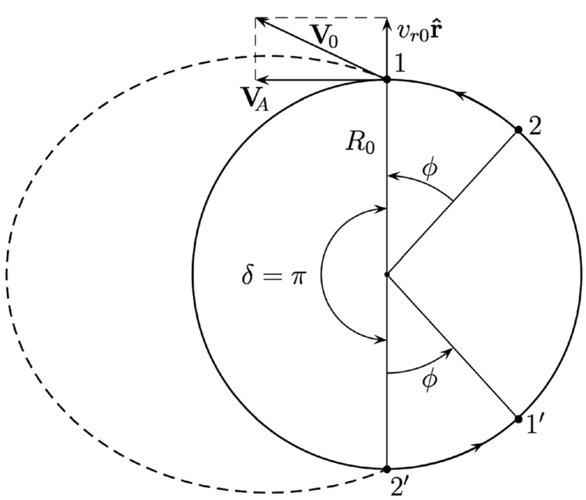

Fig. 8. Elliptical trajectory (dashed) of a sandwich sent by the first astronaut at position 1 and caught by the second astronaut at position $2'$. The astronauts move in the circular orbit and are displaced by an angle ϕ.

$$\xi = \frac{2V_0^2}{V_e^2} = 1 + \frac{v_{r0}^2 R_0}{GM_\oplus}, \tag{45}$$

where $V_e = \sqrt{2GM_\oplus/R_0}$ now means the escape velocity from an initial distance R_0 from the center of the Earth. Since $\xi > 1$, in order to compute t_{flight} we must substitute in Eq. (39) the second line of Eq. (38) (and not the first line, as in the previous example), which yields

$$t_{\text{flight}} = \sqrt{\frac{R_0^3}{2GM_\oplus}}\left(1 - \frac{\xi}{2}\right)^{-\frac{3}{2}} \times \left(\pi - \arcsin\left(\sqrt{\frac{2\xi - \xi^2}{1 + \left(\xi^2 - 2\xi\right)\cos^2\theta_0}}\sin\theta_0\right) + \sqrt{2\xi - \xi^2}\sin\theta_0\right), \tag{46}$$

where in the previous equation ξ is given in Eq. (45). If θ_0 is the smallest angle between $\mathbf{V}_0$ and $\mathbf{V}_A$, we have

$$\tan\theta_0 = \frac{v_{r0}}{V_A} \quad \Rightarrow \quad \tan^2\theta_0 = \frac{v_{r0}^2 R_0}{GM_\oplus} = \xi - 1 \quad \Rightarrow \quad \xi = 1 + \tan^2\theta_0 = \sec^2\theta_0, \tag{47}$$

where we used Eq. (45). Note that the substitution of Eq. (47) into Eq. (28) leads to $\delta = 2\arccos(0) = \pi$, as anticipated. Though $\delta = \pi$ regardless of the value of v_{r0}, this does not mean that t_{flight} is equal to half the period of the elliptical motion of the sandwich. Only for small radial oscillations this would happen, for in such a case the radial oscillations would be harmonic.

Substituting Eq. (47) into Eq. (46) it is straightforward to show that

$$t_{\text{flight}} = \frac{R_0}{V_e}\left(1 - \frac{\sec^2\theta_0}{2}\right)^{-\frac{3}{2}}\left(\pi - \arcsin\sqrt{2 - \sec^2\theta_0} + \tan\theta_0\sqrt{2 - \sec^2\theta_0}\right). \tag{48}$$

Since we are looking for the value of v_{r0} that solves the problem, it is convenient to eliminate θ_0 and ξ from the last equation in favor of v_{r0}, which can be done using Eq. (47). Doing that, and imposing the condition in Eq. (43), $\pi + \phi = \Omega t_{\text{flight}}$, the desired solution must satisfy the following transcendental equation:

$$\frac{1}{2}(\pi + \phi)\left(1 - 2\frac{v_{r0}^2}{V_e^2}\right)^{3/2} = \pi - \arcsin\sqrt{1 - \frac{2v_{r0}^2}{V_e^2}} + \frac{\sqrt{2}v_{r0}}{V_e}\sqrt{1 - \frac{2v_{r0}^2}{V_e^2}}, \tag{49}$$

which we rewrite with a simplified notation as

$$\mathcal{L}(\phi, v_{r0}/V_e) = \mathcal{R}(v_{r0}/V_e),$$

where $\mathcal{L}(\phi, v_{r0}/V_e)$ and $\mathcal{R}(v_{r0}/V_e)$ are, respectively, the left-hand side and the right-hand side of Eq. (49). This is a rather complicated equation for v_{r0}, due to its transcendental character. However, it can be solved graphically as shown in Fig. 9. We simply plot both $\mathcal{L}$—choosing, to illustrate, some values for the angular displacement ϕ—and $\mathcal{R}$ as a function of v_{r0} normalized by the escape velocity V_e, then look for the intersecting points.

It is evident from Fig. 9 that if the angular displacement ϕ between the two astronauts is increased the value of the radial component of the launching velocity of the sandwich, v_{r0}, must also be increased for astronaut 2 to catch the sandwich. Qualitatively, this can be understood as follows: as ϕ increases, the angular displacement covered by astronaut 2, namely, $\pi + \phi$, also increases so that he/she needs more time to execute uniform circular motion with angular velocity Ω. Greater times of flight are accomplished by greater values of v_{r0}.

VI. FINAL REMARKS AND CONCLUSIONS

In this work, we tried to highlight the richness of Kepler's equation in many aspects, from a historical point of view, through the many developments in both mathematics and physics that appeared in the attempts to solve this equation, to its power in solving complicated problems of planetary motion, particularly those involving the determination of time intervals in such problems. It is unbelievable that this equation was established in 1609, much before the advent of integral and differential calculus. In this sense, this equation could be taught, in principle, in very introductory courses in classical mechanics. But somewhat curiously, Kepler's equation is absent from most undergraduate textbooks. In order to illustrate explicitly how this equation can be used in solving specific problems, we chose two interesting ones which demanded the calculation of the time of flight in different situations.

In the first one, we calculated a projectile's range launched from a rotating Earth, a quite involved problem of great practical interest. We established quite general formulas for this kind of problem. For the particular case of a stationary Earth, we obtained a general formula for a projectile's range and showed explicitly that the maximum range occurs for a launching angle smaller than $\pi/4$ and that complementary angles do not lead to the same range, in contrast with what happens for small launching velocities. For the case of a rotating Earth, we showed that the projectile's range depends, indeed, on the Earth's angular velocity. In addition, in contrast to what happens for small launching velocities, two projectiles launched with velocities of the same magnitude with respect to the Earth and with the same launching angles but in opposite directions have different ranges.

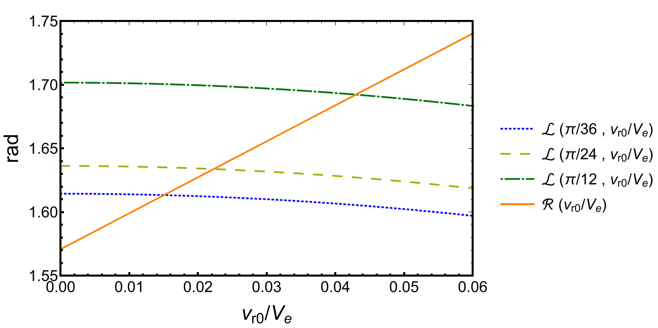

Fig. 9. Plot of Eq. (49).

In the second example, a much more academic problem, we discussed a possible way for an astronaut to send a sandwich to another astronaut moving in the same circular orbit around the Earth. For simplicity, we assumed the first astronaut impinged on the sandwich a radial impulse of magnitude mv_{r0}. It is worth emphasizing that, curious as it may seem, in this problem, no matter what the value of the radius R_0 of the circular orbit of the astronauts is, the angular displacement of the sandwich until it is caught by the second astronaut will be always equal to π. Even in this particular case, the solution is given by a transcendental equation for v_{r0}, which we solved graphically in Sec. V.

We think these two problems, as well as all the initial discussion on Kepler's equation, may serve as a nice complementary material for undergraduate courses on classical mechanics.

ACKNOWLEDGMENTS

The authors are indebted to C. E. Aguiar, Reinaldo F. de Melo e Souza, and Beto Pimentel for enlightening discussions, and to D. Szilard for help with the preparation of figures. The authors also thank CNPq and FAPERJ for partial financial support.

APPENDIX A: RELATIONSHIP BETWEEN AREAS IN SECTION II

In this appendix, we demonstrate in greater detail the relationship between the areas used to obtain Eq. (11). Area(ASQ) is the area of the region limited by segments $\overline{AS}$, $\overline{SQ}$ and arc QA in Fig. 1. We may relate Area(ASQ) to Area(ASP) as follows:

Both areas can be divided into triangles ($\triangle NSP$ and $\triangle NSQ$, respectively) and the remaining curved wedges (ANQ and ANP). In order to compare them, let us compute the ratio NP / NQ. Choosing suitable Cartesian axes $\mathcal{O}xy$ so that $\mathcal{O}x$ contains $\mathcal{O}A$, and the corresponding equations for the ellipse and the circle, respectively, we may write

$$\frac{(ON)^2}{a^2} + \frac{(NP)^2}{b^2} = 1 \;\Rightarrow\; NP = b\sqrt{1 - \frac{(ON)^2}{a^2}},$$

$$(ON)^2 + (NQ)^2 = a^2 \;\Rightarrow\; NQ = a\sqrt{1 - \frac{(ON)^2}{a^2}},$$

so that

$$\frac{\text{Area}(\triangle NSP)}{\text{Area}(\triangle NSQ)} = \frac{NP}{NQ} = \frac{b}{a}. \tag{A1}$$

For the wedges,

$$\frac{\text{Area}(ANP)}{\text{Area}(ANQ)} = \frac{\displaystyle\int_{a\cos\psi}^{a} (NP)dx}{\displaystyle\int_{a\cos\psi}^{a} (NQ)dx} = \frac{b}{a}. \tag{A2}$$

Equations (A1) and (A2) imply that the ratio b/a holds for the entire areas, as in Eq. (8).

APPENDIX B: DETAILED CALCULATIONS FOR SECTION IV B

In this appendix, we present a detailed derivation of Eqs. (36) and (37). The starting point will be Eq. (12) evaluated for ψ_0 and φ_0

$$\sin\psi_0 = \frac{\sqrt{1-\varepsilon^2}\,\sin\varphi_0}{1+\varepsilon\cos\varphi_0}. \tag{B1}$$

From Eq. (21), $\cos\varphi_0 = -\cos(\delta/2)$, we get

$$\sin\varphi_0 = \sqrt{1-\cos^2\varphi_0} = \sqrt{1-\cos^2(\delta/2)}. \tag{B2}$$

Substitution of the above equations into Eq. (B1) gives

$$\sin\psi_0 = \frac{\sqrt{(1-\varepsilon^2)(1-\cos^2(\delta/2))}}{1-\varepsilon\cos(\delta/2)}. \tag{B3}$$

Substituting Eqs. (26) and (27) into Eq. (B3), we can finally obtain $\sin\psi_0$ as a function of V_0 and θ_0, recalling that $\xi = V_0^2 R/(GM_\oplus)$

$$\sin\psi_0 = \frac{\sqrt{(1-\varepsilon^2)}\sqrt{(1-\cos^2(\delta/2))}}{1-\varepsilon\cos(\delta/2)} = \frac{\sqrt{1-\left(1+\left(\xi^2-2\xi\right)\cos^2\theta_0\right)}\sqrt{1-\frac{\left(1-\xi\cos^2\theta_0\right)^2}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}}}{1-\sqrt{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}\left(\frac{1-\xi\cos^2\theta_0}{\sqrt{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}}\right)}$$

$$= \frac{\sqrt{\left(2\xi-\xi^2\right)\cos^2\theta_0}}{\xi\cos^2\theta_0}\sqrt{\frac{\xi^2\cos^2\theta_0(1-\cos^2\theta_0)}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}} = \sqrt{\frac{\left(2\xi-\xi^2\right)}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}}\sin\theta_0,$$

which is precisely Eq. (36).

Equation (37) can be obtained by the simple trigonometric identity $\sin^2\psi_0 + \cos^2\psi_0 = 1$, so that in this case

$$\cos^2\psi_0 = 1 - \frac{\left(2\xi-\xi^2\right)\sin^2\theta_0}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0} = \frac{(\xi-1)^2}{1+\left(\xi^2-2\xi\right)\cos^2\theta_0}. \tag{B4}$$

[a]Electronic mail: fgorlando@hotmail.com
[b]Electronic mail: farina@if.ufrj.br
[c]Electronic mail: carlos.zarro@if.ufrj.br
[d]Electronic mail: pedro.terra@cp2.g12.br

[1]K. R. Symon, *Mechanics*, 3rd ed. (Addison-Wesley, Reading, 1971).
[2]S. T. Thornton and J. B. Marion, *Classical Dynamics of Particle and Systems*, 5th ed. (Thomson, Belmont, CA, 2004).
[3]N. Lemos, *Mecânica Analítica*, 2ª edição (Editora Livraria da Física, São Paulo, 2007).
[4]L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Elsevier Butterworth-Heinemann, Burlington, 1976), Vol. 1.
[5]H. Goldstein, *Classical* Mechanics, 2nd ed. (Addison-Wesley, Reading, 1980).
[6]E. Poisson and C. Will, *Gravity: Newtonian, Post-Newtonian and Relativistic* (Cambridge U.P., Cambridge, 2014).
[7]J. Kepler, Astronomia Nova, (1609), <http://www.e-rara.ch/zut/content/pageview/162861>.
[8]J. Kepler, *New Astronomy*, translated and edited by William H. Donahue (Cambridge U.P., Cambridge, 1992).
[9]F. R. Moulton, *An Introduction to Celestial Mechanics*, 2nd ed. (Macmillan, New York, 1960).
[10]T. Yoshida, "A vectorial derivation of Keplers equation," Am. J. Phys. **56**, 561–563 (1988).
[11]V. Szebehely and H. Mark, *Adventures in Celestial Mechanics*, 2nd ed. (Wiley-VCH, Weinheim, 1998).
[12]P. Colwell, *Soving Kepler's Equation over Three Centuries* (Willman-Bell, Inc., Richmond, 1993).
[13]P. Colwell, "Bessel functions and Kepler's equation," Am. Math. Mon. **99**, 45–48 (1992).
[14]C. E. Siewert and E. E. Burniston, "An exact analytical solution of Kepler's equation," Celestial Mech. **6**, 294–304 (1972).
[15]E. E. Burniston and C. E. Siewert, "Exact analytical solutions basic to a class of two-body orbits," Celestial Mech. **7**, 225–235 (1973).
[16]J. M. A. Danby and T. M. Burkardt, "The solution of Kepler's equation. I," Celestial Mech. **31**, 95–107 (1983).
[17]S. M. Perovich, M. Calasan, D. Kovac, and I. Tosic, "Concerning an analytical solution of some families of Keplers transcendental equation," AIP Adv. **6**, 035016 (2016).
[18]Y. Boetzel, A. Susobhanan, A. Gopakumar, A. Klein, and P. Jetzer, "Solving post-Newtonian accurate Kepler equation," Phys. Rev. D **96**, 044011 (2017).
[19]Walter Lewin's MIT introductory lectures on Kepler http://www.mitocw.espol.edu.ec/courses/physics/8-01-physics-i-classical-mechanics-fall-1999/video-lectures/lecture-22/.
[20]J. Barcelos Neto, *Mecânica Newtoniana, Lagrangiana & Hamiltoniana*, 1ª edição (Editora Livraria da Física, São Paulo, 2004).