Is the tautochrone curve unique?

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We show that there are an infinite number of tautochrone curves in addition to the cycloid solution first obtained by Christiaan Huygens in 1658. We begin by reviewing the inverse problem of finding the possible potential energy functions that lead to periodic motions of a particle whose period is a given function of its mechanical energy. There are infinitely many such solutions, called "sheared" potentials. As an interesting example, we show that a Pöschl-Teller potential and the one-dimensional Morse potentials are sheared relative to one another for negative energies, clarifying why they share the same oscillation periods for their bounded solutions. We then consider periodic motions of a particle sliding without friction over a track around its minimum under the influence of a constant gravitational field. After a brief historical survey of the tautochrone problem we show that, given the oscillation period, there is an infinity of tracks that lead to the same period. As a bonus, we show that there are infinitely many tautochrones.
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I. INTRODUCTION

In classical mechanics, we find essentially two kinds of problems: (i) fundamental (or direct) problems, in which the forces on a system are given and we must obtain the possible motions, and (ii) inverse problems, in which we know the possible motions and must determine the forces that caused them. Informally, we might say that in inverse problems we attempt to find the causes by analyzing the effects. A humorous way of stating the difference between these two kinds of problems is given in Bohren and Huffmann's book,[1] which states that in a direct problem, we are given a dragon and want to determine its tracks, while in an inverse problem, we start from the dragon's tracks and attempt to infer what the dragon is like.

Some historically relevant examples of inverse problems are Newton's determination of the gravitational force from Kepler's laws, and Rutherford's discovery of the atomic nucleus from the scattering of α particles from a sheet of gold foil. Indeed, inverse problems are extremely frequent even in contemporary physics. In high-energy Physics, for instance, we try to understand the fundamental interactions between elementary particles by observing the products of their scattering. Oil prospecting methods in deep waters are also inverse problems as they rely on analyzing the properties of waves (caused by small explosions at the surface of the sea) reflected by the presence of interfaces separating two different mediums within the surface of Earth.

Even in the case of a particle moving in one dimension under the influence of a conservative force, analyzing inverse problems may yield very surprising results. For instance, consider the periodic motion of a point mass in a potential well $U(x)$. Although the knowledge of $U(x)$ uniquely determines the period τ of oscillation as a function of the mechanical energy E, the opposite is not true: knowledge of $\tau(E)$ does *not* uniquely determine the potential energy that leads to the known period. In fact, as explained in Sec. II, it can be shown that a given function $\tau(E)$ allows for infinitely many potential wells that are "sheared" from one another.

This beautiful result was obtained in a quite ingenious way by Landau and Lifshitz.[2] Since then, it has been revisited by some authors using different approaches. For instance, Pippard[3] presents a nice graphical demonstration of sheared potentials, whereas Osypowski and Olsson[4] provide a different demonstration with the aid of Laplace transforms. Recent developments on this topic, and analogous quantum cases, can be found in Asorey *et al*.[5]

In this article, our main goal is to analyze inverse problems for periodic motions of a particle sliding on a frictionless track contained in a vertical plane in a uniform gravitational field. Similar to the sheared potentials, knowledge of the track shape uniquely determines the period of motion as a function of the maximum height achieved by the particle. However, as we will demonstrate, an infinity of different curves lead to the same period for each maximum height. We will show that these curves exhibit a geometrical property akin to shearing that, to our knowledge, has not been previously noted in the literature.

In particular, we are interested in the case where the period of oscillation does not depend on the maximum height (and therefore on the mechanical energy) of the system—the tautochrone. Because this term is often associated with a downward only motion, we employ the expression *round-trip tautochrone* to specify our curves of interest. It has been shown by Huygens in 1658 that the cycloid is a tautochrone curve. A direct consequence of our analysis is that this is not the sole solution of the tautochrone problem. In fact, there are infinitely many round-trip tautochrone tracks, unfolding new results for this three-century-old problem.

The remainder of the article is organized as follows. In Sec. II, we review the basic concepts of sheared potentials following Landau and Lifshitz.[2] We also provide two non-trivial examples of sheared potentials: (i) power-law potentials and (ii) a Pöschl-Teller potential and the one-dimensional Morse potentials. In Sec. III, inspired by the properties exhibited by sheared potentials, we consider the periodic motion of a particle sliding without friction on a track contained in a vertical plane under a constant gravitational force, and ask what curves yield the same period of oscillation. We show that there are an infinite number of different curves associated with each period, and provide an interpretation for the condition between these curves as a new kind of shearing. In Sec. IV, we use the previous result to find an infinite family of tautochrone curves. Finally, we make some final remarks in Sec. V.

II. BRIEF REVIEW OF SHEARED POTENTIAL WELLS

Let us consider a particle of mass m moving along the x-axis under the influence of the resultant force $\mathcal{F}(x) = -dU(x)/dx$, where $U(x)$ is a generic potential well with a single minimum. Without loss of generality, let us choose the point $x = 0$ as the minimum of the potential well such that $U(0) = 0$. Suppose the particle moves with mechanical energy E; the associated turning points are then given by the roots of the algebraic equation $E = U(x)$, which we denote by x_1 and x_2. It is worth emphasizing that x_1 and x_2 depend on E. Assuming $x_2 > x_1$, it is straightforward to use conservation of (mechanical) energy to obtain the period of oscillations as

$$\tau(E) = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}\,. \tag{1}$$

It is evident from this equation that the potential energy $U(x)$ and a given mechanical energy E uniquely determine the period of the oscillations. In other words, the function $\tau : E \to \tau(E)$ is uniquely determined by knowledge of function $U : x \to U(x)$.

In order to solve the inverse problem—given $\tau(E)$ to find the corresponding $U(x)$—it is tempting (and convenient) to regard the coordinate x as a function of the potential U. However, since the function $U(x)$ is not injective, obtaining its inverse requires defining two functions, namely, $x_L : U \mapsto x_L(U)$, for the left branch of $U(x)$ $(x < 0)$ and $x_R : U \mapsto x_R(U)$, for the right branch of $U(x)$ $(x \geq 0)$. Performing a change of variables, we recast Eq. (1) into the form

$$\begin{aligned}\tau(E) &= \sqrt{2m}\left[\int_0^E \frac{dx_R}{dU}\frac{dU}{\sqrt{E-U}} + \int_E^0 \frac{dx_L}{dU}\frac{dU}{\sqrt{E-U}}\right]\\ &= \sqrt{2m}\int_0^E \left(\frac{dx_R}{dU} - \frac{dx_L}{dU}\right)\frac{dU}{\sqrt{E-U}}\,.\end{aligned} \tag{2}$$

To get rid of the integral on the right-hand side, we multiply both sides of Eq. (2) by $dE/\sqrt{\alpha - E}$, where α is a constant parameter, and then integrate from 0 to α. On the right-hand side, we are then left with an integral in U followed by an integral in E. Changing the order of integration, which requires a subtle change in the integration limits in order to preserve the integration region in the EU-plane, we obtain

$$\int_0^\alpha \frac{\tau(E)dE}{\sqrt{\alpha - E}} = \sqrt{2m}\int_0^\alpha \left[\frac{dx_R}{dU} - \frac{dx_L}{dU}\right]dU \int_U^\alpha \frac{dE}{\sqrt{\alpha - E}\sqrt{E-U}}\,. \tag{3}$$

Fortunately, the integral over E may be exactly calculated and does not depend on U. In fact, it is straightforward to show that this integral is equal to π (it can be computed, for instance, by completing the square). As a consequence, the integration on U is immediate. From these results, and rewriting $\alpha = U$, we conclude that a given period function τ does not determine a single potential U, but rather an infinite family of potentials satisfying the relation

$$x_R(U) - x_L(U) = \frac{1}{\pi\sqrt{2m}}\int_0^U \frac{\tau(E)dE}{\sqrt{U - E}}\,. \tag{4}$$

The result tells us that, provided two potential wells $U(x)$ and $\tilde{U}(x)$ share the same width $x_R(U) - x_L(U) = \tilde{x}_R(U) - \tilde{x}_L(U)$ for every value of U, the periods of oscillation of a particle under the influence of these potentials will be described by the same function $\tau(E)$. When this property is satisfied, we say that the potential wells $U(x)$ and $\tilde{U}(x)$ are "sheared" relative to one another. However, if we further require the potential to be symmetric, so that $x_R(U) = -x_L(U)$, then the potential is uniquely determined by the period function τ.

The presentation in this section was reproduced from Landau and Lifshitz.[2] Let us illustrate the previous results in some non-trivial examples that are not usually dealt with in the literature. We will start by illustrating the shearing process for the power-law case. Then, we will show that the Morse and the Pösch-Teller potentials are related by shearing (for bounded motions).

A. Power law potentials

Sheared potentials derived from the parabolic potential well have been considered by Antón and Brun.[6] These authors obtained anharmonic motions whose periods of oscillations retain the property of being independent of the energy (isochronous oscillations).

In this subsection, we shall consider the more general case of the family of symmetric power law potentials given by $U(x) = a|x|^\nu$, with $a > 0$ and $\nu \geq 1$, whose turning points are $x_\pm = \pm(E/a)^{1/\nu}$. The period of oscillation may be computed using Eq. (1)

$$\begin{aligned}\tau(E) &= \sqrt{2m}\int_{x_-}^{x_+} \frac{dx}{\sqrt{E - a|x|^\nu}}\\ &= x_+^{-\nu/2}\sqrt{\frac{8m}{a}}\int_0^{x_+}\left[1 - \left(\frac{x}{x_+}\right)^\nu\right]^{-1/2} dx,\end{aligned} \tag{5}$$

where we have used the fact that the potential well is symmetric and that $E = ax_+^\nu$. Making a change of variables $u = (x/x_+)^\nu$ and recalling the definition for x_+, we rewrite the period as

$$\tau(E) = E^{(1/\nu - 1/2)}\frac{\sqrt{8m}}{a^{1/\nu}}\frac{I(\nu)}{\nu}, \tag{6}$$

where $I(\nu) = \int_0^1 u^{1/\nu - 1}(1-u)^{-1/2}du$ is simply a numerical factor. Although the computation may be continued to give an exact result,[2,7] it suffices for the purpose of this article to regard the period's dependence on the energy. Moreover, Eq. (6) shows that for $\nu = 2$ the exponent on E vanishes and, as a consequence, the period is independent of the energy, which is the case of the harmonic oscillator.

Let us denote by D the distance between the turning points for a given energy E; that is, $D(E) = x_+(E) - x_-(E)$. We may try to construct a sheared potential for $U(x) = a|x|^\nu$, denoted by $\tilde{U}(x)$, by defining different expressions for positive and negative x

$$\tilde{U}(x) = \begin{cases} b|x|^\nu & : x < 0\\ c|x|^\nu & : x \geq 0\end{cases} \tag{7}$$

whose difference between the two turning points corresponding to energy E is given by

$$\tilde{D}(E) = \tilde{x}_+(E) - \tilde{x}_-(E) = (E/b)^{1/\nu} + (E/c)^{1/\nu}. \qquad (8)$$

Imposing the shearing condition, $D(E) = \tilde{D}(E)$ for any E, we obtain the following condition between coefficients a, b and c:

$$\frac{1}{a^{1/\nu}} = \frac{1}{2}\left(\frac{1}{b^{1/\nu}} + \frac{1}{c^{1/\nu}}\right). \qquad (9)$$

Indeed, if we explicitly compute the period of oscillations of the particle under the influence of the potential well $\tilde{U}(x)$ by suitably adapting Eq. (6) we obtain

$$\tilde{\tau}(E) = \frac{1}{2}\left(\frac{1}{b^{1/\nu}} + \frac{1}{c^{1/\nu}}\right) E^{(1/\nu - 1/2)}\sqrt{8m}\frac{I(\nu)}{\nu}. \qquad (10)$$

Combining Eqs. (9) and (10), it becomes evident that $\tilde{\tau}(E) = \tau(E)$, as expected since we required the two potentials $U(x)$ and $\tilde{U}(x)$ to be sheared relative to each other.

B. Morse and Pöschl-Teller potentials

Let us now consider two less trivial potentials: the one-dimensional Morse potential $U_M(x)$ and the Pöschl-Teller potential $U_{PT}(x)$, given by

$$U_M(x) = U_0(e^{-2\alpha x} - 2e^{-\alpha x}) \qquad (11)$$

$$U_{PT}(x) = -\frac{U_0}{\cosh^2(\alpha x)}, \qquad (12)$$

where U_0 and α are positive constants (see Fig. 1). These two potentials have importance beyond the context of classical mechanics. For instance, the Morse potential plays an important role in chemistry while the Pöschl-Teller potential exhibits bizarre properties when treated quantum mechanically (e.g., for appropriate values of the constant U_0 this potential becomes reflectionless[8] for all values of E).

Our purpose in this subsection is to show that these two potentials are sheared relative to each other, so that they have the same period of oscillation for negative mechanical energies. With this goal, let us start by determining the turning points of a particle moving in the Morse potential with a negative mechanical energy E. These points can be obtained by setting $E = U_M(x)$. Conveniently substituting $u = e^{-\alpha x}$, this equation leads to

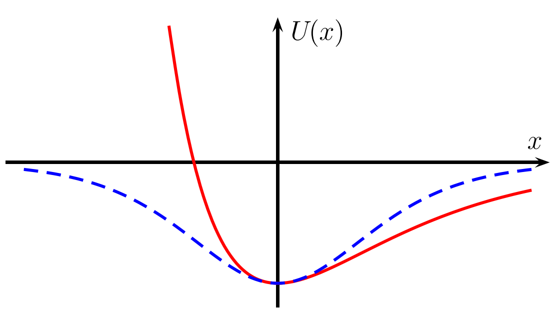

Fig. 1. The Morse (solid) and Pöschl-Teller (dashed) potentials.

$$u^2 - 2u - \frac{E}{U_0} = 0, \qquad (13)$$

which has roots $u_\pm = 1 \pm \sqrt{1 + E/U_0}$, or

$$x_\pm = \frac{1}{\alpha}\ln\left(1 \pm \sqrt{1 + E/U_0}\right). \qquad (14)$$

Thus, the distance between these two turning points, $D_M(E) = x_+(E) - x_-(E)$, for an arbitrary negative energy is given by

$$\begin{aligned} D_M(E) &= \frac{1}{\alpha}\ln\left(\frac{1 + \sqrt{1 + E/U_0}}{1 - \sqrt{1 + E/U_0}}\right) \\ &= \frac{1}{\alpha}\ln\left(\frac{\left(1 + \sqrt{1 + E/U_0}\right)^2}{-E/U_0}\right) \\ &= \frac{2}{\alpha}\ln\left(\sqrt{-\frac{U_0}{E}} + \sqrt{-\frac{U_0}{E} - 1}\right). \end{aligned} \qquad (15)$$

Let us now do the same thing for the Pöschl-Teller potential. An analogous procedure to determine the roots of the equation $E = U_{PT}(x)$ yields

$$x_\pm = \pm\frac{1}{\alpha}\operatorname{arcosh}\left(\sqrt{-U_0/E}\right). \qquad (16)$$

Therefore, the distance between these two turning points is given by

$$D_{PT}(E) = x_+ - x_- = (2/\alpha)\operatorname{arcosh}(\sqrt{-U_0/E}). \qquad (17)$$

At first glance, Eqs. (15) and (17) do not appear to coincide. However, making use of the mathematical identity $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$, it becomes evident that $D_M(E) = D_{PT}(E)$. This means, as we had anticipated, that these two potentials are sheared relative to each other for $-U_0 < E < 0$.

Because these two potentials are sheared (for $-U_0 < E < 0$), we can determine the period of oscillation using either potential (they will both give the same answer). Let us perform the calculation explicitly for the Morse potential. Using Eq. (1) and making the same change of variables as before, $u = e^{-\alpha x}$, we get

$$\begin{aligned} \tau_M(E) &= \sqrt{2m}\int_{x_-}^{x_+}\frac{dx}{\sqrt{E - U_0(e^{-2\alpha x} - 2e^{-\alpha x})}} \\ &= \sqrt{\frac{2m}{U_0\alpha^2}}\int_{u_-}^{u_+}\frac{du}{\sqrt{(E/U_0)u^2 + 2u + 1}}. \end{aligned} \qquad (18)$$

Making another change of variables $w = (u\sqrt{-E/U_0} + \sqrt{-U_0/E})$ then leads to

$$\begin{aligned} \tau_M(E) &= \sqrt{\frac{2m}{U_0\alpha^2}}\int_{w_-}^{w_+}\frac{dw}{\sqrt{w^2 - (U_0/E + 1)}} \\ &= \frac{\pi}{\alpha}\sqrt{\frac{2m}{E}}. \end{aligned} \qquad (19)$$

We leave for the interested reader the task of checking explicitly through an analogous calculation that the period associated to the Pöschl-Teller potential (also for $-U_0 < E < 0$) is indeed the same as found here.

C. General prescription for sheared potentials

After working through these examples, one might wonder whether there is a general prescription for constructing sheared potential wells. We can always begin with the symmetric potential well $U(x)$ associated with a given period function, which has $x_R(U) = -x_L(U)$. Then, in order to shear this potential we just displace the function horizontally by $\delta(U)$ to obtain new functions $\tilde{x}_R(U)$ and $\tilde{x}_L(U)$ for the right-side and the left-side branches[4,6]

$$\begin{cases} \tilde{x}_R(U) = x_R(U) + \delta(U) \\ \tilde{x}_L(U) = x_L(U) + \delta(U). \end{cases} \tag{20}$$

Consequently, the difference between the corresponding turning points is preserved for each value of U: $D(U) = \tilde{x}_R(U) - \tilde{x}_L(U) = x_R(U) - x_L(U)$. Both $\tilde{x}_R$ and $\tilde{x}_L$ must be single-valued. If this condition is met, calculating the inverse piecewise function will give a new potential $\tilde{U}(x)$ sheared from $U(x)$.

III. SHEARING TRACKS

Though not very common, sheared potential wells are very well established. Inspired by the surprising properties of these potential wells, we now turn our attention to a different problem that (as far as the authors know) has not yet been investigated. Instead of the one-dimensional motion driven by a given potential well $U(x)$, we consider the two-dimensional movement of a particle sliding along a frictionless track under a constant gravitational force. As constructing tracks is generally more feasible than supplying the conditions for obtaining an arbitrary potential well, this technique may prove useful for actual visualization of the motions under study.

We state the problem as follows: Let a particle be subject solely to a uniform gravitational field, so that the gravitational potential energy is $U(y) = mgy$, and be constrained to move along a smooth track contained in the xy-plane. Due to this constraint, the motion has only one degree of freedom. The shape of the track, described by the function $f : x \mapsto y = f(x)$, determines the net force on the particle at each point. For convenience, we choose a coordinate system so that the single minimum of the track coincides with the origin of the axes. Though the shape of the track uniquely determines the period of the motion as a function of its energy, $\tau(E)$, it is not obvious whether the knowledge of $\tau(E)$ uniquely determines the shape of the track along which the motion takes place. In other words, it is natural to ask what tracks give rise to oscillations with a given period function $\tau : E \mapsto \tau(E)$.

As will become evident later, it is convenient to use the arclength coordinate s to parametrize the track instead of the cartesian coordinate x, and henceforth we denote the shape of the track by the function $y : s \mapsto y(s)$, with $s = 0$ at the origin ($x = y = 0$). Setting the maximum height H achieved by the particle also sets the mechanical energy $E = mgH$ of the system, and the turning points s_1 and s_2 are given by the roots of the equation $y(s) = H$. From the conservation of mechanical energy, it is straightforward to obtain the corresponding period of oscillation $\tau(H)$ for an arbitrary H; this period is simply twice the time spent by the particle to move from s_1 to s_2 along the track. Denoting by v the scalar velocity of the particle at position s along the track, we have

$$\tau(H) = 2\int_{s_1}^{s_2} \frac{ds}{v} = \sqrt{\frac{2}{g}} \int_{s_1}^{s_2} \frac{ds}{\sqrt{H - y(s)}} \, . \tag{21}$$

Note that because s_1 and s_2 depend on H, the period τ is a function only of H, uniquely determined by knowledge of the track $y(s)$. Furthermore, this equation is written completely in terms of variables with a clear geometrical meaning.

We now wish to tackle the inverse problem, which is to find the function $y(s)$ that constrains the motion to a given periodic behavior $\tau(H)$. We apply the same mathematical procedure as given by Landau and Lifshitz to this new case. We invert the function $y(s)$ by means of the piecewise function split in $s_R(y)$ and $s_L(y)$ for the right and left branches of the track, respectively, and integrate using a trick analogous to that presented in Sec. II, to get

$$s_R(y) - s_L(y) = \frac{1}{\pi}\sqrt{\frac{g}{2}} \int_0^y \frac{\tau(H)dH}{\sqrt{y - H}}. \tag{22}$$

The reader may note that this result is similar to Eq. (4), with $x \to s$, $U \to y$, and $E \to H$, but we emphasize that it has a different meaning and interpretation.

Equation (22) shows that knowledge of the period function $\tau(H)$ does not uniquely determine the shape of a track, but rather its length $L(y) = s_R(y) - s_L(y)$ below every height $y > 0$. Two different tracks $y(s)$ and $\tilde{y}(s)$ will lead to motions with the same period function $\tau(H)$ if $s_R(y) - s_L(y) = \tilde{s}_R(y) - \tilde{s}_L(y)$ for every y. We call these tracks *length-sheared* relative to one another in analogy with sheared potentials. Concretely, this means that we can begin by constructing a track that has a particular functional form by using, say, measuring tape against a wall, and measure the period of oscillations of a small marble moving on it. Then we carefully re-shape the tape so as to ensure that the length of the tape under every horizontal line remains the same. This transformation will generate a new track, related to the first by length-shearing, and oscillations along this second track will have the same period as the first one for every maximum height H. This procedure is noteworthy, because a similar strategy is not generally possible with sheared potentials.

A comment is in order here. Suppose we consider only the motions of a particle that moves down a frictionless track characterized by a function $f : x \mapsto y = f(x)$ and denote by $\tau(H)$ the time spent by the particle to reach the origin at $y = 0$ after being released from rest at an arbitrary height $y = H$. Again, given the track $f(x)$, the period $\tau(H)$ is uniquely determined. However, in this case, knowledge of $\tau(H)$ would also uniquely determine the shape of the track (note that we are not talking about periodic motions here but only downward motions). This problem was first solved by Abel[9] in 1826. In 2010, Muñoz and Fernández-Anaya[10] discussed Abel's result for particular curves in which the corresponding periods are proportional to a fractional power of H. The same authors[11] (with additional collaborators) also present an illuminating set of direct and inverse problems involving beads on a frictionless rigid wire in a paper from 2011.

IV. ROUND-TRIP TAUTOCHRONES

A. Brief historical survey

Before we apply our previous result to the tautochrone problem, it is worth saying a few words about the tautochrone curve, as it played an important role in the history of classical mechanics of the 17th century. At that time, measuring latitude was relatively easy whereas measuring longitude, which was of vital importance for sea navigations, was a difficult task because it demanded a very accurate measurement of time.[12] The pendulum clock, constructed by the great Dutch physicist, mathematician, and astronomer Christiaan Huygens in 1658 (Galileo had already attempted to construct a pendulum clock but he never completed it and the patent for this invention was given to Huygens), improved the accuracy of time measurements by at least one order of magnitude, but this was not enough to guarantee a safe measurement of longitude.

With the purpose of improving maritime chronometers, Huygens started to look for an isochronous pendulum, since he knew that the simple pendulum was isochronous only for small amplitudes. A maritime chronometer constructed with an isochronous pendulum would not change the period of its oscillations even if the corresponding amplitudes changed due to a rough sea. Huygens knew that if he put lateral obstacles of appropriate shape near a simple pendulum he could achieve his purpose but, unfortunately, he was not able to find empirically the exact shape of such lateral obstacles.

Destiny then came to Huygens' aid. Blaise Pascal, the famous French physicist, mathematician, and philosopher, who had abandoned science after a religious epiphany in 1654, had an unbearable toothache in 1658 that seemed to resist any alleviating efforts. In a desperate attempt to forget the pain, Pascal decided to focus on mathematics, particularly on some problems dealing with the cycloid that the French priest Mersenne had passed to him. Soon the pain disappeared completely and Pascal interpreted this fact as a divine sign for him to go on thinking about problems involving the cycloid. He ended up solving many of Mersenne's problems and even formulated a few new ones. But instead of publishing these problems, he decided to propose a contest composed of six problems involving the cycloid.

Many important scientists of that time were encouraged to participate in that contest, including Huygens. And once he had become an expert on the cycloid, he decided to check and see, if by any chance, the cycloid would help solve the isochronous pendulum problem. Fortunately, Huygens found that the cycloid was a tautochrone—a curve on which a particle, sliding without friction and under the action of a constant gravitational force, would have a period that is independent of the height from which it was released. But he still needed to find the shape of the lateral obstacles that would make a pendulum describe a cycloidal trajectory. In other words, he had to find out what we now call the *evolute* of the cycloid. Again he tried the cycloid and once more he was successful—the evolute of the cycloid is the cycloid (shifted and out of phase)! In this particular case, Huygens was very lucky; it is not common that a curve is the evolute of itself. More details on this history can be found in Gindikin's book.[13]

Among other things, Huygens worked on trying to improve clocks for almost four decades, but his cycloidal pendulum clock, as well as his isochronous conical pendulum clock, did not succeed as maritime chronometers. However, his legacy on the developments of curves, evolutes, and involutes, which had their origins in his study of clocks, can still be seen in many different areas from differential geometry to quasicrystals.[14]

B. Obtaining the tautochrone curve

Equation (22) shows us how to construct tracks on which particles oscillate with a given period $\tau(H)$. If we impose that the track be symmetric, then it is uniquely determined. We shall now apply the previously discussed techniques to obtain the tracks on which the period does not depend on the energy; that is, solutions to the tautochrone problem. As these tracks must have upward and downward branches allowing for periodic motion, we call them *round-trip tautochrones*.

Choosing a suitable energy-independent period function $\tau(E) = \sqrt{\kappa}$, where κ is a positive constant with dimensions of time squared, and demanding that the track be symmetric allows us to compute Eq. (22) exactly

$$s(y) = \frac{1}{2\pi}\sqrt{\frac{g\kappa}{2}}\int_0^y \frac{dH}{\sqrt{y-H}} = \frac{1}{2\pi}\sqrt{\frac{g\kappa}{2}}\left(2\sqrt{y}\right), \tag{23}$$

giving

$$y = \frac{2\pi^2}{\kappa g}s^2. \tag{24}$$

The above result is in perfect analogy with the harmonic oscillator. As expected, it also corresponds to the arclength parametrization of a cycloid whose generating circle has radius $r = \kappa g/(4\pi)^2$, namely, $y = (1/8r)s^2$. The period of oscillation in terms of r and g is readily given by

$$\tau(r) = 4\pi\sqrt{\frac{r}{g}}. \tag{25}$$

Although the cycloid is indeed the single symmetric round-trip tautochrone, we may length-shear it in infinitely many ways. We will now present some unconventional solutions to the tautochrone problem, given by asymmetric tracks.

C. Half-cycloid branches

Let us begin with a symmetric cycloid whose generating circle has radius r. We first attempt an asymmetric length-sheared solution using cycloids with different generating radii. We consider for the left-side and right-side branches, respectively, $\tilde{r}_L$ and $\tilde{r}_R$, in analogy to what was presented in Sec. II A for coefficients of power-law potentials.

For both tracks to give rise to oscillations with equal periods, they must be length-sheared, so the total length of the track must be the same below every height y. The relation between the radii follows directly from applying the constraint

$$s_R(y) - s_L(y) = \tilde{s}_R(y) - \tilde{s}_L(y)\,. \tag{26}$$

From the relation $y = (1/8r)s^2$, it follows immediately that

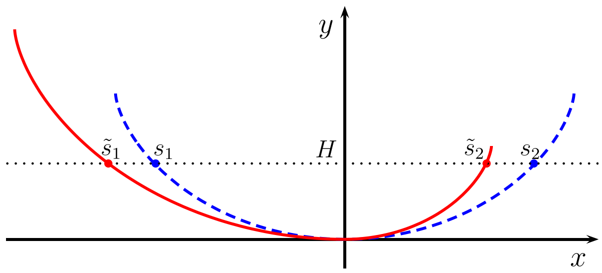

Fig. 2. Symmetric cycloid (dashed) and an isochronous asymmetric track made from different cycloids (solid).

$$2\sqrt{r} = \sqrt{\tilde{r}_R} + \sqrt{\tilde{r}_L}\,. \tag{27}$$

We present one such solution in Fig. 2.

Alternatively, we may obtain such a curve by imposing that the periods of oscillations along both the symmetric and the asymmetric tracks be the same (provided the maximum height of the motion is contained in the lower branch of the curve). The interested reader may easily verify, by means of Eq. (25), that the condition obtained is the same as in Eq. (27).

D. Completing tautochrones

Equation (22) allows for a very broad range of solutions for round-trip tautochrones that are not restricted to branches of cycloids. We may obtain such a track using a prescription similar to the one presented in Sec. II C, namely, by applying (to the original cycloidal track) equations analogous to those written in Eq. (20), but now with a length-shearing function $\delta(y)$, instead of the shearing function $\delta(U)$. With this procedure in mind, if we choose an arbitrary function for the left branch of the track, say $\tilde{y}_L(s)$, we will obtain the corresponding $\tilde{y}_R$ function for the right branch that completes the tautochrone track.

Since any track constructed in this way will be length-sheared from the original cycloidal track (which corresponds to the single symmetric tautochrone), we see that this procedure provides a method for generating as many tautochrone tracks as we want. All tracks constructed in this way will share the property of isochronous motion—all of them will lead to periodic motions that are independent of the maximum height.

An explicit example is appropriate here. Let us set the left branch of the track to the semi-cubical parabola $\tilde{y}_L = \alpha(-x)^{3/2}$ with $\alpha > 0$. We compute the arclength in order to parametrize it conveniently

$$\tilde{s}_L(y) = \frac{8}{27\alpha^2}\left[\left(\frac{9\alpha^{4/3}}{4}y^{2/3}+1\right)^{3/2} - 1\right]. \tag{28}$$

We may set the period of oscillation by adjusting the radius r of the original cycloid, $8ry = s^2$, in accordance with Eq. (25). So by imposing that the new track satisfies the length-shearing relation to this cycloid, the right-side branch of the track is given by the equation

$$\tilde{s}_R(y) = 2\sqrt{8ry} + \tilde{s}_L(y)\,. \tag{29}$$

A solution to the track having a semi-cubical parabola for its left branch is shown in Fig. 3.

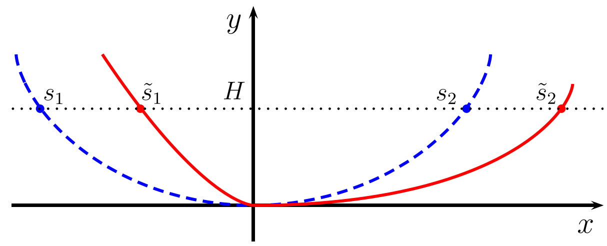

Fig. 3. Cycloid (dashed) and an isoperiodical asymmetric tautochrone having a semicubical parabola for its left branch (solid). In this example, we have set $r = 1/8$ and $\alpha = 2$.

Since we chose an arbitrary function for the left branch of the tautochrone, the time it spends on each branch will depend on the maximum height. However, the geometrical condition given by length-shearing guarantees that the difference of time the moving particle spends on the left branch (with respect to the cycloid) will be exactly compensated for on the right branch.

V. FINAL REMARKS

In the scope of one-dimensional systems in classical mechanics, we have reviewed the result that oscillations with a given period as a function of the energy correspond not to one, but to an infinite family of potential wells, which satisfy the shearing condition. Particularly, isochronous oscillations are not a property associated uniquely to the quadratic potential well (harmonic potential) but to any potential well that is sheared from it.

Then we have extended this result to the motions of a particle moving along frictionless tracks contained in a vertical plane and subject to a constant gravitational force. We have shown that by imposing a new condition—length-shearing—to the tracks, their corresponding periods, as functions of the maximum height achieved by the particle, will be the same. Applying the length-shearing condition to a cycloid, the original tautochrone discovered by Huygens in the 17th century, we have shown that it is possible to obtain an infinite family of solutions to the tautochrone problem.

The actual construction of tracks can be accomplished using a 3-D printer and is far more practical than providing the conditions for a certain potential energy $U(x)$ to drive the motion in a given period. Also, it is worth mentioning that the techniques employed in this work are accessible to undergraduate students. Hence the information presented here can be useful for enriching courses on both theoretical and experimental university-level classical mechanics.

Finally, for pedagogical reasons, we would like to outline the most important aspects of the present work:

- The study of sheared potentials, although being a well established problem, still holds a few surprises, such as the shearing relation between the Morse and Pöschl-Teller potentials.
- There is a new condition that is analogous to shearing in one-dimensional potential wells, called length-shearing, which can be applied to frictionless tracks contained in vertical planes.
- There is not one, but an infinite number of tautochrones.
- We can make a tautochrone by choosing any monotonic function $y : x \mapsto y(x)$ for the shape of one of its branches, and then computing the corresponding complementary

branch simply by imposing that they are length-sheared to the cycloid.

We leave for the interested reader the task of exploring new examples of tracks exhibiting different properties regarding the periods of their respective driven motions, further applying length-shearing.

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Plug-type Resistance Box

The plug-type resistance box, outmoded since ca. 1915, has the resistance coils connected to block of brass on the top cover. When the plug was out, the resistance between those two blocks was in the circuit. Putting in the plug shorted the resistance and gave an effective value of zero – how effective depended on how well the plug was seated and the cleanliness of the surfaces. Plugs tended to stray – this example, sold by the Chicago Apparatus Co. for $14.00 in the 1929 catalogue, has plugs from two different sources. Normally the plugs march in straight lines across the top of the box, but the designer of this series of boxes put the plugs in an arc. For $2.00 more you could get the box with glass sides so that you could see the coils. This example is in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)