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An exercise on Gauss's law for gravitation: the flat Earth model

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Abstract

We discuss the flat and hollow models of the Earth as a pedagogical example of the application of Gauss's law to the gravitational field.

Few people still believe that the Earth is flat or hollow. Or flat *and* hollow. Among the models available, some 'flat Earthers' subscribe to the notion that the Earth is a disc whose centre is at the North Pole. On the boundary of this disc, there is a thick wall of ice—the Antarctica—that prevents the waters from going over the border. Notice that there is no South Pole in this model. There are other variants of this flat Earth model, e.g. the terrestrial disc could lie on an infinite plane. The flat Earth model is a good exercise in the application of Gauss's law in a gravitational context.

Most of our students meet Gauss's law for the first time in an electrostatic context, where they learn that the flux of the electric field **E** through a closed smooth surface S is directly proportional to the net charge $q(S)$ inside the surface. The electrical flux is defined by

$$\Phi_E = \int_S \mathbf{E}\cdot\hat{\mathbf{n}}\,\mathrm{d}a, \tag{1}$$

where $\hat{\mathbf{n}}$ is the normal unit vector and da is a surface element of S. Notice that S is not necessarily closed. If the surface is closed, then Gauss's law states that (SI units)

$$\Phi_E = \oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,\mathrm{d}a = \frac{q\,(S)}{\epsilon_0}, \tag{2}$$

where ϵ_0 is the vacuum permittivity. Gauss's law is very useful when we want to evaluate the electric field associated with a highly symmetrical configuration; see [1] for details and examples. The problem we are about to discuss can also be approached by considering its electrostatic counterpart, as in [2]. Here, in order to emphasize the notion that Gauss's law holds for gravitation, in fact for any field that depends on $1/r^2$, we chose to consider its formulation for the gravitational field from the beginning.

Suppose we want to know how the gravitational field varies inside and outside the flat Earth. Gauss's law is the easiest way to answer these questions. After making the replacements $\mathbf{E} \to \mathbf{g}$ and $1/\epsilon_0 \to -4\pi\, GM\,(S)$, Gauss's law for gravitation reads

$$\Phi_g = -4\pi\, GM\,(S)\,, \tag{3}$$

where Φ_g is the flux of the gravitational field **g** through a closed smooth surface S defined by

$$\Phi_g = \oint_S \mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}a, \tag{4}$$

where G is the Newtonian gravitational constant and $M(S)$ is the mass enclosed by S. The minus sign is due to the fact that the Gaussian surface has an orientation. The unit normal vector on any point on this surface points outwards, and because **g** always points inwards, it follows that the flux is also always negative. If the mass distribution has a high degree of symmetry, e.g. spherical, cylindrical or planar, then the flux can be easily calculated

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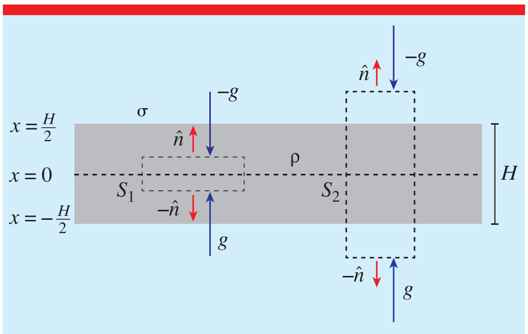

Figure 1. Flat Earth and Gaussian surfaces.

provided that we choose a Gaussian surface that respects the symmetry of the configuration at hand. If this is the case, $\mathbf{g} \cdot \hat{\mathbf{n}}$ can be factored out and the gravitational flux will be given by

$$\Phi_g = -gA, \tag{5}$$

where g is the magnitude of the field on the closed Gaussian surface whose area is A.

Though flat Earthers do not believe in gravity, we 'round Earthers' do, hence let us see how Gauss's law applies to the flat Earth model. First of all we must realize that the terrestrial disc is really a highly flattened cylinder. This means that if a is the radius of the cylinder and H is its height, or better, its thickness, then the condition $a \gg H$ holds. If we additionally agree to consider points far away from the border of the cylinder, then planar symmetry applies, provided that the mass distribution ρ is uniform or a function of the thickness of the (flat) Earth only. Notice that this means that the field is perpendicular to the mass distribution. For simplicity, we will also suppose that ρ is uniform and its numerical value equal to the mean density of the spherical Earth. If we adopt these assumptions, we can adopt the cylindrical surfaces S_1 and S_2 as Gaussian surfaces, as sketched in Figure 1. Consider for example S_1. The flux through this surface is

$$\Phi_g = -2gA_{\text{top}}, \tag{6}$$

and the mass enclosed by S_1 is

$$M(S_1) = \rho A_{\text{top}} x. \tag{7}$$

Gauss's law then leads to

$$g(x) = 4\pi\, G\rho\, x, \tag{8}$$

for the field inside the distribution. For the evaluation of the field outside the distribution we make use of S_2. Then, proceeding in the same way, we find $\Phi_g = -2gA_{\text{top}}$, but this time the mass enclosed is $M(S_2) = \rho H A_{\text{top}}$. It follows from Gauss's law that in magnitudes outside the mass distribution $g = 2\pi\, G\sigma$, where we have defined $\sigma = \rho H$ as the mean surface density of the (flat) Earth. We can collect these results taking into account their domain of validity and direction in the formula given below (notice that the field is continuous on the surface of the distribution)

$$g(x) = -\begin{cases} 2\pi\, G\sigma, & x \leqslant -\frac{H}{2}; \\ -4\pi\, G\rho x, & -\frac{H}{2} \leqslant x + \leqslant \frac{H}{2};, \\ -2\pi\, G\sigma, & x \geqslant +\frac{H}{2} \end{cases} \tag{9}$$

And what if the Earth is flat and hollow? In this case, the reader can easily verify that the gravitational field reads

$$g(x) = \begin{cases} 4\pi\, G\sigma, & x \leqslant -\frac{H}{2}; \\ 0, & -\frac{H}{2} \leqslant x \leqslant \frac{H}{2};. \\ -4\pi\, G\sigma, & x \geqslant \frac{H}{2} \end{cases} \tag{10}$$

The flat Earth model or the hollow flat Earth model must reproduce the measured value of the gravitational acceleration on the surface of the Earth. In the case of the former model, it is reasonable to ask ourselves how its thickness H compares to the radius R of the spherical Earth [2]. The mass enclosed by the Gaussian cylinder will be $M = \rho\, AH$ where, as mentioned before, ρ is the mean density of the spherical Earth. On the surface of the (flat) Earth

$$g = 2\pi G\, \rho H. \tag{11}$$

Since g must be equal to the gravitational acceleration on the surface of the (spherical) Earth we have

$$g = \frac{G\, M_{\text{Earth}}}{R^2}, \tag{12}$$

where R is the mean radius of the terrestrial sphere. But $M_{\text{Earth}} = \rho\,(4\pi/3)\,R^3$, hence we can also write

$$g = G\frac{4\pi\, R\rho}{3}. \tag{13}$$

Setting equation (11) equal to equation (13) it follows that

$$H = \frac{2}{3}R = \frac{2}{3}6371\,\text{km} \approx 4247\,\text{km}. \tag{14}$$

Notice that this result does not depend on the mean density of the Earth. If the Earth were a 'lighter' or 'heavier' planet this result would still hold. As a consistency check the reader can infer the value of ρ from the measured value of g on the surface of the Earth.

The thickness of the flat Earth can also be inferred from experimental data. Suppose we measure g on the surface of the Earth and find 9.807 ms^{-2}, and from rock samples we conclude that the mean mass density ρ is 5515 kgm^{-3}. Then from equation (11) we can write

$$H = \frac{g}{2\pi\, G\rho} = \frac{9.807}{2\pi\, 6.673\times 10^{-11}\times 5.515}\text{m} = 4\,241\,208.91\ \text{m} \approx 4241\ \text{km}, \quad (15)$$

in good agreement with the theoretical result given by equation (14). Suppose flat Earth engineers decide to bore a tunnel to communicate with flat Earthers on the other side of the world. A tunnel boring machine working around the clock can progress more or less 15 meters per day. A simple calculation will show that the engineers will need 775 years to reach the other side of the Earth!

Notice that in order to take full advantage of Gauss's law as a tool for evaluating the gravitational field, we have considered in practice unlimited planar mass distributions. If, for instance, our models are substituted by really finite discs, then Gauss's law, though still holding in a general way, loses its effectiveness as a calculation tool and other methods, e.g. gravitational potential expansion techniques, are more advisable. Infinite mass and charge distributions can quickly lead us into mathematical and physical trouble, though they are useful as approximations [3].

Flat Earth models have a long history and there are many variants of them. Here we have discussed just two of these models. More information on flat Earth theories can be found at [4]. See also Sir Patrick Moore's 'Can you speak Venusian?' for a delightful point of view on this subject [5]. If the reader wants the real thing, try for example [6].

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