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The motion of two masses coupled to a finite mass spring

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Abstract

We discuss the classical motion of a finite mass spring coupled to two pointlike masses fixed at its ends. A general approach to the problem is presented and some general results are obtained. Examples for which a simple elastic function can be inferred are discussed and the normal modes and normal frequencies obtained. An approximation procedure to the evaluation of the normal frequencies in the case of uniform elastic function and mass density is also discussed.

1. Introduction

The motion of one or two masses linked by a massless spring constrained to move on a straight line and without friction is analysed in several introductory and undergraduate mechanics textbooks, see for example [1, 2]. In the case of two arbitrary masses, the two-body problem is solved by a reduction to the problem of a single body oscillating with an angular frequency equal to $\sqrt{K_e/\mu}$, where K_e is the spring constant and μ is the reduced mass of the system, and to the motion of the centre of mass of the system the velocity of which is constant if no external forces are present. Moreover, since only the masses make contributions to the kinetic energy and to the total linear momentum, the mechanical energy conservation theorem and the linear momentum conservation theorem can be applied without much ado. The forces acting on the masses are due to the spring deformation at the extremities where the masses are fixed to. This is the reason why Newton's third law of motion cannot be directly applied to the masses. We are forced to consider in more detail the mechanism of interaction between the two masses and in particular their interaction with the extremities of the spring to which each one of them is fixed to. However, due to the fact that the spring is massless we can also state that at any given moment of time the sum of those forces is zero. Then, in an equivalent way, we can think that the masses move under the action of the force that one mass exerts on the other, thereby complying with the third law in such a way that we can ignore the existence of the spring.

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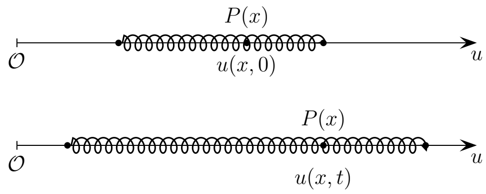

Figure 1. The motion of a point of the spring with respect to an inertial frame is described by the coordinate $u(x, t)$. Given a point P of the spring, we associated with it the parameter x. This association is independent of the dynamical state of the spring.

The correction to the frequency for the case where one of the springs is held fixed and the mass m of the spring, though not zero, is much less than the mass M fixed to the oscillating end is well known. In this case in order to get the angular frequency up to first order, we can consider the massless spring and replace the mass of the oscillating body by an effective mass that is equal to $M + m/3$, see for example [1], see also [3] and references therein.

In this paper we will consider a more general situation. We will consider the problem of two arbitrary masses, say M_1 and M_2, fixed to a spring of arbitrary finite mass m. The effects caused by the propagation of the spring deformation along the spring length will be taken into account. Solutions to particular situations such as the ones described above will be considered as appropriate limits of a less particular solution. We believe that the approach we take here may be of some pedagogical value for advanced students and instructors as well.

2. The equations of motion of the system

We begin by establishing the equation of motion for the finite mass spring along a single spatial dimension. In order to do so we introduce an auxiliary parameter x that will help us to describe the properties of the spring such as, for example, its tension or its density at a given point. With this aim in mind, let us consider the spring in a non-deformed state and denote by ℓ its natural length. Now we define a one-to-one correspondence between the spring viewed as a one-dimensional smooth matter distribution and the closed interval $[0, \ell]$ in such a way that $x = 0$ corresponds to the left end of the spring and $x = \ell$ to its right end, see figure 1. To an arbitrary point P on the spring corresponds a point $x \in [0, \ell]$. The parameter x must not be viewed as a regular spatial coordinate. This parameter can be thought of, if we wish, as an internal degree of freedom of the spring and it is not subject to the transformations associated with the one-dimensional Galilean group, for instance, non-relativistic boosts or translations. Were the string made up of N discrete masses labelled by a discrete index j running from 1 to N, this index would have played a role analogous to x. We assume that the correspondence established here holds for any state of motion of the spring, exactly as in the case of the discrete model. Let it be now an inertial reference frame $\mathcal{S}$ and a suitable coordinate system and let us suppose that the spring moves along the u-axis in such a way that the position of a point of the spring with respect to $\mathcal{S}$ is given by the function $u(x, t)$, see figure 1. The tension T at a point of a spring is given by [4]

$$T(x,t) = \kappa(x)\frac{\partial\eta(x,t)}{\partial x}, \tag{1}$$

where $\kappa(x)$ is the elastic function of the spring which, on physical grounds, we suppose to be always positive and $\eta(x, t)$ is its deformation function. If the spring is in motion then the deformation can be written as

$$\eta(x,t) = u(x,t) - u(0,t) - x. \tag{2}$$

Therefore, we can write

$$T(x,t) = \kappa(x)\left(\frac{\partial u(x,t)}{\partial x} - 1\right). \tag{3}$$

It is not hard to see that $\kappa(x) > 0$ for any $x \in [0, \ell]$. At a given point, the force that the right portion of the spring exerts on the left portion will be $T(x,t)$ and conversely the force that the left portion of the spring exerts on the right portion will be $-T(x,t)$. Consider now an element of the spring determined by x and $x + \mathrm{d}x$. The resultant force acting on this element is

$$\begin{aligned} \mathrm{d}F(x,t) &= T(x+\mathrm{d}x, t) - T(x,t) \\ &= \frac{\partial}{\partial x}\left[\kappa(x)\left(\frac{\partial u(x,t)}{\partial x} - 1\right)\right]\mathrm{d}x. \end{aligned} \tag{4}$$

If $\rho(x)$ is the linear mass density of the spring then, after applying Newton's second law of motion to the element of mass $\rho(x)\,\mathrm{d}x$, we obtain

$$\frac{\partial}{\partial x}\left[\kappa(x)\left(\frac{\partial u(x,t)}{\partial x} - 1\right)\right] = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{5}$$

Equation (5) controls the motion of the spring. It can be simplified by introducing the variable

$$\xi(x,t) = u(x,t) - x. \tag{6}$$

Then the equation of motion of the spring will read

$$\frac{\partial}{\partial x}\left[\kappa(x)\frac{\partial \xi(x,t)}{\partial x}\right] = \rho(x)\frac{\partial^2 \xi(x,t)}{\partial t^2}. \tag{7}$$

A word of caution: though $\xi(x,t)$ is related to the deformation of the spring, it does not represent this deformation directly.

Note that the equation of motion of the massive spring in the form given by equation (5) or (7) is invariant under Galilean transformations. In fact, if we go from the inertial system $\mathcal{S}$ to the inertial system $\mathcal{S}'$ that moves with velocity V with respect to $\mathcal{S}$, the following evident relations hold:

$$u'(x,t) = u(x,t) + Vt, \tag{8}$$

$$\frac{\partial u'(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial x}, \tag{9}$$

$$\frac{\partial^2 u'(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{10}$$

In this sense, the Galilean invariance of the equation of motion of the spring is manifest in accordance with the fact that this equation derives from a straightforward application of the principles of Newtonian mechanics.

Let us now consider the coupled masses. Let us model them by means of two pointlike particles, one of mass M_1 coupled to the left end ($x = 0$) of the spring and the other of mass M_2 coupled to the right end ($x = \ell$). Making use of equation (7) and Newton's second and third laws, we can write the equations of motion of the masses as

$$M_1\frac{\partial^2 \xi(0,t)}{\partial t^2} = \kappa(0)\frac{\partial \xi(0,t)}{\partial x}, \tag{11}$$

$$M_2\frac{\partial^2 \xi(\ell,t)}{\partial t^2} = -\kappa(\ell)\frac{\partial \xi(\ell,t)}{\partial x}. \tag{12}$$

The complete solution of these equations and of equation (7) demands that we prescribe the initial conditions

$$\xi(x, t = 0) = \varphi(x) - x, \tag{13}$$

$$\frac{\partial \xi(x, t = 0)}{\partial t} = \psi(x), \tag{14}$$

where $\varphi(x) = u(x, 0)$ and $\psi(x) = \partial u(x, t = 0)/\partial t$ describe the initial position and velocity of the points of the spring.

Our aim is to obtain a general solution $\xi(x, t)$—or $u(x, t)$—to the problem and therefore describe an arbitrary state of motion of the system, i.e. the two pointlike masses plus the massive string.

3. General solution of the equations of motion

We begin by solving equation (7) by the method of separation of variables, that is, we look for a solution of the form

$$\xi(x, t) = X(x)T(t) \tag{15}$$

that satisfy also the boundary conditions given by equations (11) and (12). Substituting equation (15) into (7) and introducing the separation constant $-\lambda$, we have

$$\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} + \lambda T(t) = 0, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right] + \lambda\rho(x)X(x) = 0. \tag{17}$$

Equation (11) imposes a boundary condition on equation (17). In order to obtain this boundary condition, we substitute equation (15) into equation (11) and write

$$M_1 X(0)\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} = \kappa(0)T(t)\frac{\mathrm{d}X(0)}{\mathrm{d}x}, \tag{18}$$

and, taking into account equation (16), we obtain (the prime indicates derivative with respect to x)

$$\kappa(0)X'(0) = -\lambda M_1 X(0). \tag{19}$$

In the same way, substituting equation (15) into (12) and combining with equation (16), we obtain the condition

$$\kappa(\ell)X'(\ell) = \lambda M_2 X(\ell), \tag{20}$$

Let us now show that the eigenvalue λ cannot assume negative values. Suppose that we have an eigenfunction $X(x)$ corresponding to a particular eigenvalue λ. Consider the following identity which can be derived after an integration by parts and use of equations (17), (19) and (20):

$$\int_0^\ell \kappa(x)X'^2(x)\,\mathrm{d}x = \lambda\left[M_1 X^2(0) + M_2 X^2(\ell)) + \int_o^\ell \rho(x)X^2(x)\,\mathrm{d}x\right]. \tag{21}$$

Since the left-hand side is always non-negative and the bracket on the right-hand side is always positive, we conclude that λ is non-negative.

The null eigenvalue is physically acceptable and has a special meaning. The reason is that $\lambda = 0$ is common to all springs regardless of their mass density, elastic function and the masses of the bodies fixed to their extremities. Note that the corresponding eigenfunction (the zero mode) can be obtained in a general form. This particular mode is not associated with an oscillatory motion of the spring. In fact, for $\lambda = 0$ the temporal function has the form

$$T(t) = \beta_0 + \alpha_0 t, \tag{22}$$

where α_0 and β_0 are constants. On the other hand, equation (17) for $\lambda = 0$ yields

$$\frac{\mathrm{d}X(x)}{\mathrm{d}x} = \frac{\gamma}{\kappa(x)}, \tag{23}$$

where γ is an integration constant. Boundary conditions as expressed by equations (11) and (12), or equivalently equations (19) and (20), demand $\gamma = 0$ so that $X(x) = b =$ constant. The eigenfunction corresponding to this eigenvalue is then

$$\begin{aligned}\xi_0(x,t) &= b(\beta_0 + \alpha_0 t)\\ &= x_0 + V_0 t,\end{aligned} \tag{24}$$

where we have introduced the new constants x_0 and V_0. It is clear that this solution corresponds to a uniform motion of the entire system (masses plus spring) with a common velocity V_0. The zero-mode motion is related to Galilean boosts and may be added to any other solution of the problem if questions about the Galilean invariance are an issue.

Finally, let us consider the case of positive λ. Setting $\lambda = \omega^2$ for convenience, we write the solutions of equation (16) as

$$T_n(t) = A_n \cos(\omega_n t + \phi_n), \tag{25}$$

where n is a positive integer, ω_n is the nth frequency eigenvalue indexed in crescent order ($\omega_1 < \omega_2 < \omega_3 \cdots$) and A_n and ϕ_n are constants. The nth eigensolution to equation (7) corresponding to the nth eigenfrequency is

$$\xi_n(x,t) = X_n(x) T_n(t). \tag{26}$$

These modes represent the oscillatory modes of the system. The general solution can be written as

$$\xi(x,t) = x_0 + V_0 t + \sum_{n=1}^{\infty} X_n(x) T_n(t). \tag{27}$$

Consequently, in terms of the function $u(x,t)$, the general solution will be given by

$$u(x,t) = x_0 + x + V_0 t + \sum_{n=1}^{\infty} X_n(x) T_n(t). \tag{28}$$

The next step is the explicit determination of the spectrum of eigenfrequencies. This is, however, a hard task to perform and in principle it can be accomplished only if we also know explicitly the elastic function $\kappa(x)$. As mentioned before, the zero mode is the only mode that does not depend on the form of $\kappa(x)$.

4. The orthogonality of the eigenfunctions

Before dealing with concrete examples, let us consider a little bit more some of the formal aspects of our problem. Equations (19) and (20) can be read as boundary conditions for equation (17). Therefore, only for certain values of λ, there will be solutions to this equation. The reader will immediately recognize that we are dealing with a Sturm–Liouville system. Let us then consider two different eigenvalues, say λ_m and λ_n and their respective eigenfunctions $X_m(x)$ and $X_n(x)$. These eigenfunctions satisfy the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)\frac{\mathrm{d}X_m(x)}{\mathrm{d}x}\right] + \lambda_m \rho X_m(x) = 0 \tag{29}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)\frac{\mathrm{d}X_n(x)}{\mathrm{d}x}\right] + \lambda_n \rho X_n(x) = 0. \tag{30}$$

As usual, we multiply the first equation by X_n and the second by X_m, subtract one from the other and after simple additional manipulations, we end up with

$$(\lambda_m - \lambda_n)\rho(x)X_m(x)X_n(x) + \frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)\left(X_n(x)\frac{\mathrm{d}X_m(x)}{\mathrm{d}x} - X_m(x)\frac{\mathrm{d}X_n(x)}{\mathrm{d}x}\right)\right] = 0. \tag{31}$$

Integrating this last equation over the domain $[0, \ell]$ and taking into account the boundary conditions given by equations (19) and (20), we obtain, after some simplifications,

$$(\lambda_m - \lambda_n)\left[\int_0^l \rho(x)X_m(x)X_n(x)\,\mathrm{d}x + M_2X_m(l)X_n(l) + M_1X_m(0)X_n(0)\right] = 0. \tag{32}$$

At this point, we define a scalar product that will be convenient for our purposes. Let the functions $f(x)$ and $g(x)$ be defined in the closed interval $[0, l]$. Then, by definition, their scalar product is

$$\langle f, g\rangle = \int_0^l \rho(x)f(x)g(x)\,\mathrm{d}x + M_2 f(l)g(l) + M_1 f(0)g(0). \tag{33}$$

With this definition for the scalar product, we can consider the eigenfunctions corresponding to different eigenvalues as an orthonormal set of eigenfunctions, i.e.

$$\langle X_m, X_n\rangle = \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots. \tag{34}$$

By making use of the initial conditions and the above orthonormality condition, the determination of the constants A_n and ϕ_n that appear in equation (25) and, therefore, in the general solution can be carried out in a systematic way. For the zero mode, for instance, we have

$$X_0(x) = \frac{1}{\sqrt{M_1 + M_2 + m}}. \tag{35}$$

5. Conservation laws

Linear momentum and mechanical energy conservation theorems can be proved under quite general conditions. The former depends on the fact that the system is isolated and the latter depends also on the fact that the internal forces can be considered as conservatives. Let us first consider the linear momentum of the system. Our goal will be to determine explicitly the contribution of the massive spring to the total linear momentum.

The linear momentum due to the two pointlike masses is given by

$$P_{\text{masses}} = M_1\frac{\partial u(0,t)}{\partial t} + M_2\frac{\partial u(\ell,t)}{\partial t}. \tag{36}$$

Making use of equations (11) and (12), we can recast the total time derivative of P_{blocks} into the form

$$\frac{\mathrm{d}P_{\text{blocks}}}{\mathrm{d}t} = k(0)\frac{\partial\xi(0,t)}{\partial x} - k(\ell)\frac{\partial\xi(\ell,t)}{\partial x}. \tag{37}$$

On the other hand, we can integrate equation (7) over the domain $[0, \ell]$ to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^{\ell} \rho(x)\,\xi(x,t)\,\mathrm{d}x = k(\ell)\,\frac{\partial\xi(\ell,t)}{\partial x} - k(0)\frac{\partial\xi(0,t)}{\partial x}. \tag{38}$$

Taking this result into equation (37), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\mathrm{masses}} + \int_0^{l} \rho(x)\frac{\partial\xi(x,t)}{\partial t}\,\mathrm{d}x\right) = 0. \tag{39}$$

Defining the linear momentum of the spring by

$$P_{\mathrm{spring}} = \int_0^{l} \rho(x)\frac{\partial u(x,t)}{\partial t}\,\mathrm{d}x = \int_0^{l} \rho(x)\frac{\partial\xi(x,t)}{\partial t}\,\mathrm{d}x, \tag{40}$$

we see that the total linear momentum of the system $P_{\mathrm{blocks}} + P_{\mathrm{spring}}$ is conserved. The total linear momentum can be rewritten in the form

$$P_{\mathrm{total}} = M_1\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} + M_2\frac{\mathrm{d}u_2(t)}{\mathrm{d}t} + \int_0^{l} \rho(x)\frac{\partial u(x,t)}{\partial t}\,\mathrm{d}x, \tag{41}$$

where $u_1(t) \equiv u(0,t)$ and $u_2(t) \equiv u(\ell,t)$. In terms of $\xi(x,t)$, we have

$$P_{\mathrm{total}} = M_1\frac{\mathrm{d}\xi_1(t)}{\mathrm{d}t} + M_2\frac{\mathrm{d}\xi_2(t)}{\mathrm{d}t} + \int_0^{l} \rho(x)\frac{\partial\xi(x,t)}{\partial t}\,\mathrm{d}x. \tag{42}$$

From equation (27) or (28), we can rewrite the total linear momentum in the form

$$P_{\mathrm{total}} = (M_1 + M_2 + m)V_0 + \sum_{n=1}^{\infty}\left[M_1X_n(0) + M_2X_n(\ell) + \int_0^{\ell} \rho(x)X_n(x)\,\mathrm{d}x\right]\dot{T}(t). \tag{43}$$

This expression can be rewritten in the form (the dot indicates derivative with respect to the time)

$$P_{\mathrm{total}} = (M_1 + M_2 + m)V_0 + \sum_{n=1}^{\infty}\langle X_m, X_0(x)\rangle\dot{T}((t). \tag{44}$$

Since X_0 and X_n are orthogonal, we see that only the zero mode contributes to the total linear momentum

$$P_{\mathrm{total}} = (M_1 + M_2 + m)V_0. \tag{45}$$

From this result we see that the constant V_0 is the velocity of the centre of mass of the system, as expected.

We now consider the mechanical energy of the system. The kinetic energy of the pointlike masses is given by

$$T_1 + T_2 = \frac{1}{2}M_1\left(\frac{\partial u(0,t)}{\partial t}\right)^2 + \frac{1}{2}M_2\left(\frac{\partial u(\ell,t)}{\partial t}\right)^2, \tag{46}$$

which evidently is not *per se* a conserved quantity because the pointlike masses exchange energy with the spring. It follows that in order to have conservation of the mechanical energy, it is mandatory that any variation of the kinetic energy of the blocks be compensated by a variation of the energy of the spring, kinetic, potential or both. Keeping this in mind, we derive (46) with respect to the time to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_1 + T_2) = M_1\frac{\partial u(0,t)}{\partial t}\frac{\partial^2 u(0,t)}{\partial t^2} + M_2\frac{\partial u(\ell,t)}{\partial t}\frac{\partial^2 u(\ell,t)}{\partial t^2}. \tag{47}$$

Combining this result with equations (11) and (12), we can eliminate the masses of the pointlike bodies and write

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_1+T_2)=\kappa(0)\frac{\partial u(0,t)}{\partial t}\left[\frac{\partial u(0,t)}{\partial x}-1\right]-\kappa(\ell)\frac{\partial u(\ell,t)}{\partial t}\left[\frac{\partial u(\ell,t)}{\partial x}-1\right]. \tag{48}$$

We can recast this equation into a more useful form if we first multiply equation (5) by $\partial u(x,t)/\partial t$ to obtain

$$\rho(x)\frac{\partial u(x,t)}{\partial t}\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left[\kappa(x)\left(\frac{\partial u(x,t)}{\partial x}-1\right)\frac{\partial u(x,t)}{\partial t}\right]$$
$$-\frac{1}{2}\kappa(x)\frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial x}-1\right)^2. \tag{49}$$

Then, integrating this result over the interval $[0,\ell]$, we will have

$$\kappa(\ell)\left(\frac{\partial u(\ell,t)}{\partial x}-1\right)\frac{\partial u(\ell,t)}{\partial t}-\kappa(0)\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t}=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l\frac{1}{2}\rho(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2\mathrm{d}x$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l\frac{1}{2}\kappa(x)\left(\frac{\partial u(x,t)}{\partial x}-1\right)^2. \tag{50}$$

Substituting this last equation into equation (48), it follows after one more integration that

$$E=T_1+T_2+\int_0^l\frac{1}{2}\rho(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2\mathrm{d}x+\int_0^l\frac{1}{2}\kappa(x)\left(\frac{\partial u(x,t)}{\partial x}-1\right)^2\mathrm{d}x. \tag{51}$$

This equation expresses the conservation of the total mechanical energy of the system. The first three terms on the rhs of equation (51) represent the kinetic energy of the pointlike masses and of the massive spring; the last term represents the potential energy of the spring. We can rewrite equation (51) in terms of $\xi(x,t)$:

$$E=T_1+T_2+\int_0^\ell\frac{1}{2}\rho(x)\left(\frac{\partial \xi(x,t)}{\partial t}\right)^2\mathrm{d}x+\int_0^\ell\frac{1}{2}\kappa(x)\left(\frac{\partial \xi(x,t)}{\partial x}\right)^2\mathrm{d}x, \tag{52}$$

which turns out to be more useful in some applications.

Proceeding as in the case of the total linear momentum, we can write the total energy in terms of the general solution. The result is

$$E_{\text{total}}=\frac{1}{2}\left(M_1+M_2+m\right)V_0^2+\sum_{n=1}^{\infty}\left[M_1X_n(0)+M_2X_n(\ell)+\int_0^\ell\rho(x)X_n(x)\,\mathrm{d}x\right]\dot{T}_n(t)$$
$$+\sum_{n,m=1}^{\infty}\left[M_1X_n(0)X_m(0)+M_2X_n(\ell)X_m(\ell)\right.$$
$$\left.+\int_0^\ell\rho(x)X_n(x)X_m(x)\,\mathrm{d}x\right]\dot{T}_n(t)\dot{T}_m(t)$$
$$+\int_0^\infty\kappa(x)\sum_{n,m=1}^{\infty}X_n'(x)X_m'(x)T_n(t)T_m(t)\,\mathrm{d}x. \tag{53}$$

The last term representing the potential energy of the spring can be suitably rewritten with the help of the following identity:

$$\int_0^\ell\kappa(x)X_n'(x)X_m'(x)\,\mathrm{d}x=\lambda_n\left[M_1X_n(0)X_m(0)+M_2X_n(\ell)X_m(\ell)\right.$$
$$\left.+\int_0^\ell\rho(x)X_n(x)X_m(x)\,\mathrm{d}x\right], \tag{54}$$

which can be easily proved. The final result is

$$E_{\text{total}} = \frac{1}{2}(M_1 + M_2 + m)V_0^2 + \sum_{n=1}^{\infty}\left[M_1 X_n(0) + M_2 X_n(\ell) + \int_0^{\ell}\rho(x)X_n(x)\,\mathrm{d}x\right]\dot{T}_n(t)$$
$$+ \sum_{n,m=1}^{\infty}\left[M_1 X_n(0)X_m(0) + M_2 X_n(\ell)X_m(\ell)\right.$$
$$\left. + \int_0^{\ell}\rho(x)X_n(x)X_m(x)\,\mathrm{d}x\right]\dot{T}_n(t)\dot{T}_m(t)$$
$$+ \int_0^{\infty}\kappa(x)\sum_{n,m=1}^{\infty}\langle X_n(x)X_m(x)\rangle T_n(t)T_m(t)\,\mathrm{d}x. \tag{55}$$

Taking into account the orthonormality relation, we can recast the total energy in a more illuminating form

$$E_{\text{total}} = \frac{1}{2}(M_1 + M_2 + m)V_0^2 + \sum_{n=1}^{\infty}\left[\dot{T}_n^2(t) + \omega_n^2 T_n^2\right] \tag{56}$$

or

$$E_{\text{total}} = \frac{1}{2}(M_1 + M_2 + m)V_0^2 + \sum_{n=1}^{\infty}\omega_n^2\left(\alpha_n^2 + \beta_n^2\right). \tag{57}$$

This last result shows that the total energy of the system can be decomposed into a sum of energies, each one associated with a normal mode. Moreover, we can see that in order to excite two or more frequencies of comparable amplitudes, it is necessary to supply the mode with the highest frequency with a greater amount of external energy.

6. Solution for $\rho(x) = 0$

When the spring is massless, the motion of the two pointlike masses is easily obtained by reducing the two-body problem to the motion of a single effective body about a centre of force [2]. Here, we try to obtain those solutions by making use of equations (16) and (17).

Firstly, note that taking $\rho(x) = 0$ does not eliminate the possibility of having eigenvalues different from zero. It only means that the spatial eigenvalue equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right] = 0, \tag{58}$$

whose solution is

$$X(x) = C\int_0^x \frac{\mathrm{d}x'}{\kappa(x')} + X(0). \tag{59}$$

The position of a point of the spring is then given by

$$u(x,t) = x_0 + x + V_0 t + \left[C\int_0^x \frac{\mathrm{d}x'}{\kappa(x')} + X(0)\right]A\cos(\omega t + \phi). \tag{60}$$

Defining the usual spring constant K_e by

$$K_e^{-1} = \int_0^{\ell}\frac{\mathrm{d}x}{\kappa(x)}, \tag{61}$$

with the help of equation (59), we obtain

$$C = K_e[X(\ell) - X(0)]. \tag{62}$$

Making use of the boundary conditions in the form given by equations (19) and (20), we have

$$K_e X(\ell) - (K_e - \omega^2 M_1) X(0) = 0, \tag{63}$$

$$(K_e - \omega^2 M_2) X(\ell) - K_e X(0) = 0. \tag{64}$$

In order to have a non-trivial solution, the determinant associated with this linear system must be zero, that is

$$M_1 M_2 \omega^4 - K_e (M_1 + M_2)\omega^2 = 0. \tag{65}$$

It follows that the allowed eigenfrequency, as expected, is given by

$$\omega_1 = \sqrt{\frac{K_e}{\mu}}, \tag{66}$$

where

$$\mu = \frac{M_1 M_2}{M_2 + M_1} \tag{67}$$

is the reduced mass of the system, mass 1 plus mass 2. This eigenfrequency and the zero-mode frequency $\omega_0 = 0$ are the only allowed frequencies of the system when the spring is massless. Making use of the equation of our definition of the scalar product, we can easily calculate the constant C that normalizes the eigenfunction. The result is

$$C = \frac{K_e}{\sqrt{\mu}}. \tag{68}$$

7. Solution for $\kappa(x)$ and $\rho(x)$ uniform

We now turn our attention to an important special case. When the elastic function of the spring and its density are uniform, it is possible to solve analytically the equation of motion, i.e. the wave equation that describes the system and clearly interpret the solutions. Defining

$$v^2 = \kappa/\rho, \tag{69}$$

and setting $\lambda = \omega^2$ in order to be in accordance with the standard notation, equations (16) and (17) read

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} + \omega^2 T = 0, \tag{70}$$

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + q^2 X = 0, \tag{71}$$

where

$$\omega = qv. \tag{72}$$

The boundary conditions, equations (11) and (12) or (19) and (20), applied to this particular case lead to

$$M_1 X(0) q^2 v^2 + \kappa X'(0) = 0, \tag{73}$$

$$M_2 X(\ell) q^2 v^2 - \kappa X'(\ell) = 0. \tag{74}$$

The general solution for the spatial part is

$$X(x) = A\cos qx + B\sin qx. \tag{75}$$

The allowed eigenvalues are determined by the linear algebraic system

$$M_1 q v^2 A + kB = 0, \tag{76}$$

$$(M_2 q v^2 \cos q\ell + \kappa \sin q\ell)A + (M_2 q v^2 \sin q\ell - \kappa\cos q\ell)B = 0, \tag{77}$$

whose characteristic equation is

$$\tan q\ell = \frac{\rho q}{\mu q^2 - \frac{\rho^2}{M_1+M_2}}. \tag{78}$$

To illustrate the discussion, let us consider the situation for which the density of the spring is very small. In this case, the mass of the spring can be neglected. Making the necessary approximations to equation (78), we obtain

$$q \approx \sqrt{\frac{\rho}{\mu\ell}}. \tag{79}$$

The angular frequency is given by equation (72) and in this case it leads to

$$\omega \approx \sqrt{\frac{\kappa}{\mu\ell}} = \sqrt{\frac{K_e}{\mu}}. \tag{80}$$

As expected, for the case at hand we obtain a non-trivial limit as the mass density tends to zero. Note that κ/ℓ was identified with the usual elastic constant of the spring. Note also that the speed of the wave does depend on the density of the spring and tends to infinity as the mass density tends to zero. It is precisely due to this fact that in this approximation, it is possible to replace the real forces by forces between the two pointlike masses obeying Newton's action and reaction principle discussed in the introduction. To investigate the next order correction to the angular frequency, we consider for simplicity the case where one of the point masses, say M_1, is infinite and the total mass of the point particles $M_1 + M_2$ is also infinite. This situation corresponds to the case where one of the extremities of the spring is fixed to a wall. Adding one more term to the expansion of $\tan q\ell$ in (78), we obtain the following quartic equation for q:

$$\frac{1}{3}q^4 + \frac{1}{\ell^2}q^2 - \frac{\rho}{M_2\ell^3} = 0, \tag{81}$$

whose physical solution for $m/M_2 \ll 1$ is given by

$$q = \sqrt{\frac{\rho}{M_2\ell}} - \frac{1}{6}\sqrt{\frac{\ell\rho^3}{M_2^3}}. \tag{82}$$

Consequently, we will have

$$\omega = \left(\sqrt{\frac{\rho}{M_2\ell}} - \frac{1}{6}\sqrt{\frac{\ell\rho^3}{M_2^3}}\right)\sqrt{\frac{\kappa}{\rho}}. \tag{83}$$

A little bit more of simple algebra allows us to write

$$\omega \approx \sqrt{\frac{\kappa}{\ell\left(M_2 + \frac{1}{3}m\right)}}, \tag{84}$$

a well-known result, see for example [1, 3].

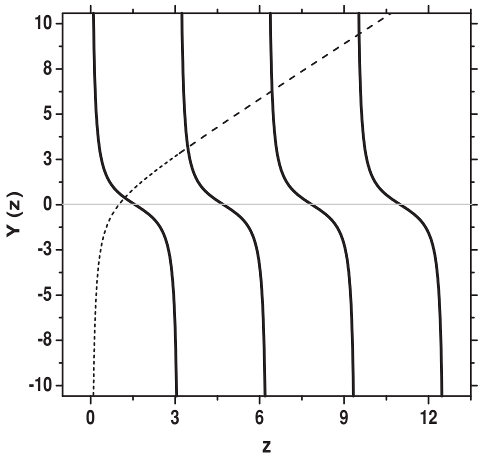

Figure 2. The dashed curve is the graphical representation of the rhs of equation (85) for given m, M_1 and M_2.

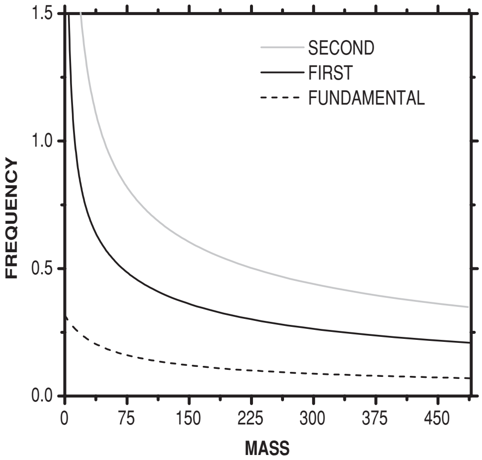

Figure 3. The curves show the behaviour of the frequency of the first three modes of the system as a function of the mass of the spring. One of the masses, M_1, is infinite, and $M_2 = 10$ mass units.

8. The angular eigenfrequencies

In order to investigate a general solution of equation (78), let us define the variable $z = q\ell$ and write the characteristic equation (78) in the form

$$\cot z = \frac{\mu z}{m} - \frac{m}{Mz}, \tag{85}$$

where $M = M_1 + M_2$. In figure 2, we plot the lhs and rhs of equation (85) separately for representative values of μ, M and m. The solutions of the characteristic equation are determined by the intersection points. It is easily seen that there is an infinite number of eigenfrequencies, one in each open interval $(n\pi, (n+1)\pi)$, where n is a non-negative integer. The lowest eigenfrequency lies in the interval $(0, \pi)$. The lowest eigenfrequency is the only

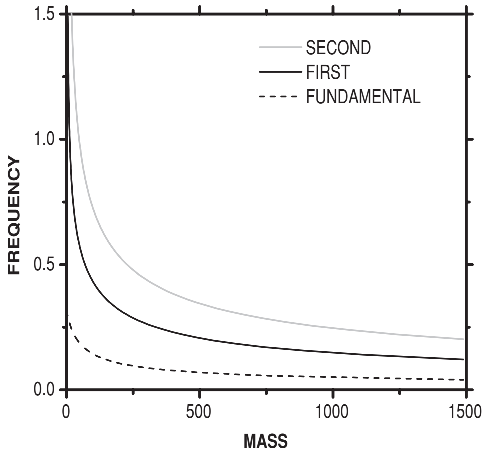

Figure 4. The curves show the behaviour of the frequency of the first three modes of the system as a function of the mass of the spring. One of the masses, M_1, is infinite, and $M_2 = 10$ mass units.

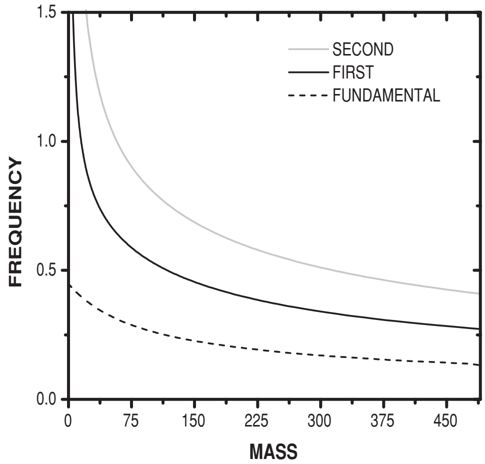

Figure 5. The curves show the behaviour of the frequency of the first three modes of the system as a function of the mass of the spring. The sum of the pointlike masses is finite, $M_1 + M_2 = 10$ mass units.

one that remains finite when the mass of the spring tends to zero. All other eigenfrequencies tend to be infinite, and this means that they are increasingly harder to excite. For $n \gg 0$, the highest eigenfrequencies can be approximately described by the simple formula $z_n = n\pi$. Then, we can write

$$\omega_n \approx n\pi\sqrt{\frac{\kappa}{\rho\ell^2}} = n\pi\sqrt{\frac{K_e}{m}}. \tag{86}$$

In order to obtain an analytical approximate solution for the eigenfrequencies, we solve equation (85) for m to obtain

$$m = \frac{1}{2\tan z}(-M \pm \sqrt{(M^2 + 4(\tan^2 z)\mu M)})z, \tag{87}$$

where the plus sign must be used if $x \in (n\pi, n\pi + \pi/2)$ and the minus sign if $z \in (n\pi + \pi/2), (n+1)\pi$. Now we define $w = \sqrt{m}$ and make use of the Bürmann–Lagrange theorem [5] to express the inverse function in the series form. The result is

$$z = \frac{1}{\sqrt{\mu}}\left[w + \left(-\frac{1}{6\mu} + \frac{1}{2M}\right)w^3 + \left(\frac{11}{360\mu^2} - \frac{1}{12\mu M} - \frac{1}{8M^2}\right)w^5 + \cdots\right]. \tag{88}$$

Consider only the first term of this series. Then, it is easily seen that

$$q \approx \frac{1}{\ell}\sqrt{\frac{m}{\mu}}. \tag{89}$$

In this case, the angular frequency is

$$\omega_0 \approx \sqrt{\frac{\kappa}{\mu}}. \tag{90}$$

Let us consider the first correction to this result which means to take into account the term in w^3 in the inverted series. Then, it follows that

$$q \approx \frac{1}{\ell}\sqrt{\frac{m}{\mu}}\left[1 + \left(-\frac{1}{6\mu} + \frac{1}{2M}\right)m\right]. \tag{91}$$

The angular frequency is then

$$\omega_0 \approx \sqrt{\frac{\kappa}{\mu}}\left[\left(1 - \frac{1}{6\mu} + \frac{1}{2M}\right)m\right]. \tag{92}$$

We can also express the other eigenfrequencies ($n > 0$) in a series form by again using the Bürmann–Lagrange theorem. The result up to the fourth power in the mass of the spring is

$$\omega_n = n\pi\sqrt{\frac{K_e}{m\ell}}\left[1 + \frac{m}{\mu n^2\pi^2} - \frac{m^2}{\mu^2 n^3\pi^3} + \left(\frac{2}{\mu^3 n^6\pi^6} - \frac{1}{3\mu^3 n^4\pi^4} + \frac{1}{\mu^2 n^4\pi^4 M}\right)m^3 \right.$$
$$\left. + \left(-\frac{5}{\mu^4 n^8\pi^8} + \frac{4}{3\mu^4 n^6\pi^6 M} - \frac{4}{\mu^3 n^6\pi^6 M}\right)m^4\right]. \tag{93}$$

Figures 3–5 show the behaviour of the first three lowest eigenfrequencies as a function of the mass of the spring m for a particular choice of the sum of the pointlike masses $M_1 + M_2$. In figures 3 and 4, one of the masses is infinite and the other is finite. This means that one of the ends of the spring is coupled to a fixed wall. In figure 5 both pointlike masses are finite. As physically expected, when the mass of the spring goes to zero the higher modes become harder and harder to excite and the fundamental mode tends to a fixed value.

9. Concluding remarks

In this paper we discussed the classical mechanics of a spring of finite mass coupled to two arbitrary pointlike masses fixed at its ends. A general approach to the problem was attempted and some general results such as the conservation of linear momentum and energy were obtained. We have also shown that the physical problem leads to an example of a Sturm–Liouville system. The detailed study of this problem is heavily dependent on the explicit knowledge of the elastic function $\kappa(x)$. The special case for which the elastic function and the mass density are uniform was discussed, and an approximation procedure to the evaluation of the normal frequencies was put forward and tested. In the limiting case of a massless spring, we have focused our attention on the motion of the fixed masses M_1 and M_2 and considered the spring as a way of transmitting the interaction between them. With respect to the wave motion of the spring, we observe that the result $\sqrt{K_e/\rho}$ is the velocity of the wave only if the velocity of the centre of mass of the system is zero. If this velocity is V with respect to some suitable reference frame then, according to the Galilean rule, the velocity of pulse propagation will be $V = \sqrt{K_e/\rho}$.

At the moment, the study of possible equivalence between motion in a single mode of the massive spring and simple harmonic motion and possible quantization of the system is under way.

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