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NOTE

A note on the Lorentz force, magnetic charges and the Casimir effect

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Abstract
We show that in order to account for the repulsive Casimir effect in the parallel-plate geometry in terms of the quantum version of the Lorentz force, it is possible to introduce virtual surface densities of magnetic charge and currents. The quantum version of the Lorentz force expressed in terms of the correlators of the electric and magnetic fields for planar geometries then yields the Casimir pressure correctly.

1. Introduction

Since its prediction by Casimir in 1948, the Casimir effect [1] has been the object of an increasing number of theoretical and experimental investigations. This is due to growing recognition of its fundamental importance in quantum field theory and also in elementary particle physics, cosmology and condensed matter physics as well as its practical and decisive role in nanotechnology. For an introduction to this remarkable effect, see for example [2]; for an updated review of the recent research and applications see [3] and references therein.

Even in simple examples the Casimir interaction can exhibit surprising features. Some time ago Gonzales [4], among other things, correctly pointed out that an alternative computation of the Casimir force between two perfectly conducting plates can be carried out starting from the consideration of the Lorentz force acting on the plates. The reason is that in this context the quantum version of the Lorentz force is the only force that could act on a metallic plate and therefore one should be able to obtain Casimir's result from this physical fact. Here we will develop this point of view further and consider its consequences when applied to Boyer's variant of the standard Casimir effect in which one of the conducting plates is replaced by a magnetically permeable one [5]. We will show that when applied to this particular case Gonzales' conception of the Casimir interaction leads to the introduction of virtual magnetic charges and currents.

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2. The standard Casimir effect

In order to state our point of view we begin by considering the standard experimental set-up proposed by Casimir, which consists of two infinite perfectly conducting parallel plates kept at a fixed distance a from each other. We will choose the coordinate axis in such a way that the OZ direction is perpendicular to the plates. One of the plates will be placed at $z = 0$ and the other one at $z = a$. Classically, the Lorentz force per unit area on, say, the conducting plate at $z = a$ is given by

$$\vec{f}_e = \frac{1}{2}\sigma_e \vec{E} + \frac{1}{2c}\vec{K}_e \times \vec{B}, \tag{1}$$

where σ_e is the electric charge density and $\vec{K}_e$ is the electric current surface density. The boundary conditions on the electric and the magnetic fields on the plate are as follows: the tangential components E_x and E_y of the electric field and the normal component B_z of the magnetic field must be zero on the plate. Under the boundary conditions imposed on the field at $z = a$, however, it is easily seen that the resultant classical Lorentz force is perpendicular to the conducting plate. We expect the quantum version of equation (1) to show the same feature and ultimately to be the source of the Casimir pressure between the two conducting plates. The electric charge and current densities on the conducting plate are related to the fields through

$$\hat{n} \cdot \vec{E} = 4\pi\sigma_e, \tag{2}$$

$$\hat{n} \times \vec{B} = \frac{4\pi}{c}\vec{K}_e, \tag{3}$$

where $\hat{n}$ is the normal to the plate under consideration. The quantum version of equation (1) reads

$$\langle \vec{f}_e \rangle_0 = \frac{1}{8\pi}\langle \vec{E}^2 - \vec{B}^2 \rangle_0 \hat{n}, \tag{4}$$

and can be obtained by combining the vacuum expectation value of equations (1)–(3). In order to proceed—from now on we depart from [4]—we need to evaluate the vacuum expectation value of the quantum operators $E_i(\vec{r}, t)E_j(\vec{r}, t)$, $B_i(\vec{r}, t)B_j(\vec{r}, t)$ and $E_i(\vec{r}, t)B_j(\vec{r}, t)$. The evaluation of these correlators depends on the specific choice of boundary conditions. A regularization recipe is also necessary because these objects are mathematically ill-defined. Regularization recipes vary from the relatively simple cut-off method employed by Casimir himself [1] to the sophisticated and mathematically elegant generalized zeta-function techniques (see [6] for an introduction to these techniques). Here we will make use of the results and direct the reader to the relevant references. The electric field correlators for a pair of perfectly conducting plates are given by [9–11, 13]

$$\langle E_i(\vec{r}, t)E_j(\vec{r}, t)\rangle_0 = \left(\frac{\pi}{a}\right)^4 \frac{2}{3\pi}\left[\frac{(-\delta^{\parallel}_{ij} + \delta^{\perp}_{ij})}{120} + \delta_{ij}F(\xi)\right], \tag{5}$$

where δ_{ij} is the Kronecker delta, $\delta^{\parallel}_{ij} := \delta_{ix}\delta_{jx} + \delta_{iy}\delta_{jy}$ and $\delta^{\perp}_{ij} := \delta_{iz}\delta_{jz}$. The function $F(\xi)$ with $\xi := \pi z/a$ is defined by

$$F(\xi) := -\frac{1}{8}\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\frac{1}{2}\cot(\xi), \tag{6}$$

and its expansion about $\xi = \xi_0$ is given by

$$F(\xi) \approx \tfrac{3}{8}(\xi - \xi_0)^{-4} + \tfrac{1}{120} + \mathrm{O}[(\xi - \xi_0)]^2. \tag{7}$$

Notice that due to the behaviour of $F(\xi)$ near $\xi_0 = 0, \pi$, strong divergences control the behaviour of the correlators near the plates. The corresponding magnetic field correlators are

$$\langle B_i(\vec{r}, t)B_j(\vec{r}, t)\rangle_0 = \left(\frac{\pi}{a}\right)^4 \frac{2}{3\pi}\left[\frac{(-\delta^{\parallel}_{ij} + \delta^{\perp}_{ij})}{120} - \delta_{ij}F(\xi)\right]. \tag{8}$$

A direct evaluation also shows that the correlators $\langle E_i(\vec{r},t)B_j(\vec{r},t)\rangle_0$ are zero. For the purposes of calculation it is convenient to consider a third conducting plate placed perpendicularly to the OZ axis at $z = \ell$. Consider the plate at $z = a$. The Lorentz force per unit area on its left side ($\hat{n} = -\hat{z}$) reads

$$\langle f_z^L\rangle_0 = -\frac{1}{8\pi}\langle \vec{E}^2 - \vec{B}^2\rangle_0 \approx -\frac{3}{16\pi^2(z-a)^4} - \frac{\pi^2}{240\,a^4}, \tag{9}$$

where we have also made use of equation (7). On the other hand, after simple modifications in equations (5), (8) and (7) the Lorentz force on the right side of the plate ($\hat{n} = \hat{z}$) reads

$$\langle f_z^R\rangle_0 = \frac{1}{8\pi}\langle \vec{E}^2 - \vec{B}^2\rangle_0 \approx \frac{3}{16\pi^2(z-a)^4} + \frac{\pi^2}{240\,(\ell-a)^4}. \tag{10}$$

Adding the forces on both sides of the plate and setting $\ell \to \infty$ we obtain the well known result

$$\langle f_z\rangle_0 = \langle f_z^R\rangle_0 + \langle f_z^R\rangle_0 = -\frac{\pi^2}{240\,a^4}. \tag{11}$$

The minus sign means that the probe plate at $z = a$ is attracted towards the other one at the origin.

3. The repulsive version of the standard Casimir effect

Let us now consider an alternative set-up to the standard one in which a perfectly conducting plate is placed at $z = 0$ and a perfectly permeable one is placed at $z = a$. This set-up was analysed for the first time by Boyer in the context of stochastic electrodynamics [5], a kind of classical electrodynamics that includes the zero-point electromagnetic radiation, and leads to the simplest example of a repulsive Casimir interaction. For alternative evaluations see [7, 8]. The boundary conditions now are:

(a) the tangential components E_x and E_y of the electric field as well as the normal component B_z of the magnetic field must vanish on the surface of the plate at $z = 0$;
(b) the tangential components of B_x and B_y of the magnetic field as well as normal component E_z of the electric field must vanish on the surface of the plate at $z = a$.

From the classical point of view the perfectly permeable plate at $z = a$ poses a problem when we apply equation (1) to it. This is so because due to the boundary conditions this time the Lorentz force on either side of the plate is *parallel* to the permeable plate and therefore the resultant Lorentz force will also be parallel to the plate. This is a puzzling feature if we wish to describe the Casimir interaction between the plates through the Lorentz force. Things can be mended, however, if we allow for a virtual surface magnetic charge density σ_m and a virtual magnetic charge current surface density $\vec{K}_m$. In this case the modified Lorentz force per unit area on the permeable plate reads

$$\vec{f}_m = \frac{1}{2}\sigma_m\vec{B} - \frac{1}{2c}\vec{K}_m \times \vec{E}. \tag{12}$$

The charge and current surface densities are related to the fields through

$$\hat{n}\cdot\vec{B} = 4\pi\sigma_m, \tag{13}$$

$$\hat{n}\times\vec{E} = -\frac{4\pi}{c}\vec{K}_m. \tag{14}$$

It is easily seen that the modified Lorentz force given by equation (12) combined with the boundary conditions on the permeable plate yields on either side of the plate a force perpendicular to the plate, as must be the case. Proceeding as above we now have

$$\langle \vec{f}_m\rangle_0 = \frac{1}{8\pi}\langle \vec{B}^2 - \vec{E}^2\rangle_0\hat{n}. \tag{15}$$

For Boyer's set-up the relevant correlators were evaluated in [12, 13]. The results are

$$\langle E_i(\vec{r},t)E_j(\vec{r},t)\rangle_0 = \left(\frac{\pi}{a}\right)^4 \frac{2}{3\pi}\left[\left(-\frac{7}{8}\right)\frac{(-\delta_{ij}^{\parallel}+\delta_{ij}^{\perp})}{120} + \delta_{ij}\, G(\xi)\right], \tag{16}$$

$$\langle B_i(\vec{r},t)B_j(\vec{r},t)\rangle_0 = \left(\frac{\pi}{a}\right)^4 \frac{2}{3\pi}\left[\left(-\frac{7}{8}\right)\frac{(-\delta_{ij}^{\parallel}+\delta_{ij}^{\perp})}{120} - \delta_{ij}\, G(\xi)\right], \tag{17}$$

where

$$G(\xi) = -\frac{1}{8}\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\frac{1}{2\sin(\xi)}. \tag{18}$$

Near $\xi = 0$ the function $G(\xi)$ behaves as

$$G(\xi) = \tfrac{3}{8}\xi^{-4} - \tfrac{7}{8}\tfrac{1}{120} + \mathrm{O}(\xi^2), \tag{19}$$

but near $\xi = \pi$ its behaviour is slightly different

$$G(\xi) = -\tfrac{3}{8}(\xi-\pi)^{-4} + \tfrac{7}{8}\tfrac{1}{120} + \mathrm{O}[(\xi-\pi)^2]. \tag{20}$$

Again, a direct calculation shows that $\langle E_i(\vec{r},t)B_j(\vec{r},t)\rangle_0 = 0$ for this case also.

To obtain the Casimir force it is convenient to replace the third plate at $z = \ell$ by a permeable one. It is not hard to see that if we do so we can use equations (5) and (8) in the region between the two permeable plates with small modifications. The force on the left side ($\hat{n} = -\hat{z}$) of the permeable plate at $z = a$ then reads

$$\langle f_{m,z}^L\rangle_0 \approx -\frac{3}{16\pi^2(z-a)^4} + \frac{7}{8}\times\frac{\pi^2}{240\,a^4}, \tag{21}$$

and the force on the right side is

$$\langle f_{m,z}^R\rangle_0 \approx \frac{3}{16\pi^2(z-a)^4} + \frac{\pi^2}{240\,(\ell-a)^4}. \tag{22}$$

As before we add the forces on each side and set $\ell \to \infty$ to obtain the repulsive Casimir force per unit area

$$\langle f_{m,z}\rangle_0 = \langle f_{m,z}^L\rangle_0 + \langle f_{m,z}^R\rangle_0 = \frac{7}{8}\times\frac{\pi^2}{240\,a^4}, \tag{23}$$

in agreement with Boyer [5]. Notice that this time the force per unit area is repulsive. Notice also that in both cases the divergent pieces cancel out; these cancellations yield finite Casimir energies [9, 11–13].

4. Conclusions

As we can see the introduction of virtual magnetic charges and currents can account for Boyer's variant of the Casimir effect in terms of the quantum version of the Lorentz force. Of course, for other geometries and boundary conditions, for example a perfectly conducting cube, the Casimir force can be repulsive and can be accounted for by the usual virtual electric charges and currents. On the other hand, it is plausible to state that whenever we have an ideal magnetically permeable wall as one of the confining surfaces a qualitative analysis of the interaction between the zero-point electromagnetic fields and this confining surface, which is modelled by appropriate boundary conditions, will show the need to introduce virtual magnetic charges and currents. The introduction of these charges and currents avoids the need to construct a model of the Casimir interaction based on an appropriate distribution of amperian currents, a considerably harder task.

As a final remark we observe that in the framework of the usual cavity quantum electrodynamics, the result given by equation (23) is obtained by first evaluating the confined vacuum renormalized energy followed by a variation of the volume of the confining region. It can easily be shown, however, that in terms of partition functions and free energies, Boyer's set-up is mathematically equivalent to the difference between two standard set-ups, one with the distance between the plates equal to $2a$ and the other one with the distance between the plates equal to a (see [7]). This can also be easily proved for the Casimir pressure at zero and finite temperature.

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References

[1] Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
See also
Applequist T, Chodos A and Freund P G O (ed) 1987 *Modern Kaluza–Klein Theories* (Reading, MA: Addison-Wesley)
Special section: Hendrik Casimir (1909–2000) 2001 *Eur. J. Phys.* **22** 441–69
[2] Milonni P W 1994 *The Quantum Vacuum: an Introduction to Quantum Electrodynamics* (Boston, MA: Academic)
See also
Lamoreaux S K 1999 *Am. J. Phys.* **10** 850
[3] Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
See also
Milton K A 2001 *The Casimir Effect: Physical Manifestations of Zero-Point Energy* (Singapore: World Scientific)
Mostepanenko V M 1997 *The Casimir Effect and its Applications* (Oxford: Clarendon)
[4] Gonzales A E 1985 *Physica* A **131** 228
[5] Boyer T H 1974 *Phys. Rev.* A **9** 2078
[6] Reuter M and Dittrich W 1985 *Eur. J. Phys.* **6** 33
Elizalde E and Romeo A 1991 *Am. J. Phys.* **59** 711
[7] Santos F C, Tenorio A and Tort A C 1999 *Phys. Rev.* D **60** 105022
[8] da Silva J C, Matos Neto A, Placido H Q, Revzen M and Santana A E 2001 *Physica* A **292** 411
[9] Lütken C A and Ravndal F 1985 *Phys. Rev.* A **31** 2082
[10] Bordag M, Robaschik D and Wieczorek E 1985 *Ann. Phys., NY* **165** 192
[11] Barton G 1990 *Phys. Lett.* B **237** 559
[12] Cougo-Pinto M V, Farina C, Santos F C and Tort A C 1999 *J. Phys. A: Math. Gen.* **32** 4463
[13] Farina C, Passos Sobrinho J J and Tort A C 2000 Maxwell stress tensor and the Casimir effect *Preprint* hep-th/0007201