Part I: Cosmology Basics

[Ryden chap. 2 to 4]

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Chapter 2
Fundamental Observations

- Olber's Paradox
- Homogeneity and isotropy
- Hubble's (Lemaître's) Law
- Cosmologist Particle Book
- The CMB
Olber's Paradox

Why is the night sky dark?

- Infinite and static universe → bright sky!
- Solution 1: universe has finite size
- Solution 2: universe has a finite age
Olber's Paradox (2)

- Parenthesis: Luminosity vs. Brightness vs. Intensity vs. Flux
  - **Luminosity** \((L)\) = total energy / time; emitted or received
    - It is a *property* of the source; does **not** depend on distance
  - **Intensity** \((I)\) = **Brightness** = energy / (time x det. area x solid ang.)
    - It is a *property* of the source; does **not** depend on distance
    - Specific Intensity \((I_\nu)\) = \(I / \text{(unit frequency)}\)
  - **Flux** \((f)\) = Luminosity / (4\(\pi\) distance\(^2\))
    - depends on distance

\[
I = \int_0^\infty I_\nu d\nu
\]

\[
f(r) = \frac{L}{4\pi r^2}
\]
Olber's Paradox (3)

- Let's compute the sky brightness
  - Let $n$ be the average # density of stars
  - Let $L$ be their average luminosity ($= \text{energy/time}$)

$$\text{flux} : f(r) = \frac{L}{4\pi r^2}$$

The intensity differential $dJ$ is:

$$dJ(r) = \frac{L}{4\pi r^2} \cdot n \cdot r^2 dr = \frac{nL}{4\pi} dr$$

$$J = \int_{r=0}^{\infty} dJ = \frac{nL}{4\pi} \int_{0}^{\infty} dr = \infty$$

Where did we go wrong?
Olber's Paradox (4)

- Stars have finite (angular) size → they obstruct stars behind them
  - $J$ no longer $\infty$; instead $J \leftrightarrow$ average surface brightness of a star

- $L$ and/or $n$ may depend on distance
  - We would need $L n \propto r^x$, $x < -1$

- Universe could have finite size and/or age
  - Cutoff in the integral

- Flux might not go down as $1/r^2$
  - Due to non-euclidean geometry
  - Due to redshift / expansion
Homogeneity & Isotropy

- The universe is* homogeneous and isotropic on large scales
  - Minimum scale ~ 100 Mpc
  - * – our observations are consistent with this hypotheses
- Isotropy & homogeneity independent
  - Isotropy around every point → homogeneity

- Copernican Principle:
  - we don't live in a special location in the universe

- Cosmological Principle:
  - on sufficiently large scales, the properties of the Universe are the same for all observers
Homogeneity & Isotropy (2)

- Isotropy & homogeneity independent
  - Isotropy around every point → homogeneity

Anisotropic & Homogeneous

Anisotropic & Inhomogeneous
Homogeneity & Isotropy (3)
The Hubble's Law

- The Doppler allows us to measure radial velocities with high precision;

\[ z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} - 1 = \frac{v}{c} + \mathcal{O}\left(\frac{v}{c}\right)^2 \]
The Hubble's Law (2)

- Lemaître (and later Hubble)* found out that galaxies are, in average, receding from us;
  - The redshift ($z$) is linear with distance
  - The velocity is approx. also linear with distance
  - * Stigler's law of eponymy: "No scientific discovery is named after its original discoverer."

\[
v = H_0 r \quad (z \ll 1)
\]

\[
z = \frac{H_0}{c} r + O \left( \frac{v}{c} \right)^2
\]
$h = 0.72 \pm 0.03 \pm 0.07$ Freedman et al. (Hubble Key Project)

$z \approx 0.1$

$10^9$ light-years

$2 \times 10^9$ light-years

Hubble's data

Riess et al astro-ph/9410054
Quasar Spectra (different z’s)

- B2 1128+31  $z=0.178$
- PKS 1217+02  $z=0.240$
- 4C 73.18  $z=0.302$
- B2 1208+32A  $z=0.389$

Wavelength (angstroms)
The Hubble's Law (3)

- Hubble's Law does not violate the copernican principle!
  - Isotropic and homogeneous expansion produces Hubble's law for all observers

\[
H_0 = (69 \pm 2) \frac{\text{km}}{\text{s Mpc}}
\]
The Hubble's Law (4)

- We can describe such an expansion by a time-dependent scale factor $a(t)$
  - Inhomogeneity $\rightarrow a(t, r)$
  - Anisotropy (shear) $\rightarrow a(t, \theta, \varphi)$ or $\{a(t), b(t), c(t)\}$
The Hubble's Law (5)

- If galaxies are receding from us, were we once together?
  - Simplest first assumption: $H(t) = \text{const} = H_0$
  - This implies ALL galaxies were together at the SAME time

\[
t_0 = \frac{r}{v} = \frac{r}{H_0 r} = H_0^{-1}
\]

\[H_0^{-1} = (14.2 \pm 0.4) \text{ Gyr}\]

- This is the base of the Big-Bang model
- The above calculation ignores gravity
  - Gravity pulls galaxies in ans slows expansion with time

\[H(t) > H_0 \text{ in the past } \Rightarrow t_0 < 14.2 \text{ Gyr}\]
4 types of particle are important in cosmology:

- Photons, “baryons” (protons+neutrons+electrons), neutrinos, and dark matter
  - $e^-$ mass $\ll$ proton mass
  - Neutrinos are almost always “free-streaming”

<table>
<thead>
<tr>
<th>particle</th>
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<th>charge</th>
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<tr>
<td>proton</td>
<td>$p$</td>
<td>938.3</td>
<td>+1</td>
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<tr>
<td>neutron</td>
<td>$n$</td>
<td>939.6</td>
<td>0</td>
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<tr>
<td>electron</td>
<td>$e^-$</td>
<td>0.511</td>
<td>-1</td>
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<tr>
<td>neutrino</td>
<td>$\nu_e, \nu_\mu, \nu_\tau$</td>
<td>?</td>
<td>0</td>
</tr>
<tr>
<td>photon</td>
<td>$\gamma$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>dark matter</td>
<td>?</td>
<td>?</td>
<td>0</td>
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</tbody>
</table>
In thermal equilibrium the energy density of photons is given by the blackbody spectrum.

\[
\varepsilon(f) df = \frac{8\pi h}{c^3} \frac{f^3 df}{\exp(hf/kT) - 1}
\]

The total energy and number density are:

\[
\varepsilon = \alpha T^4 \quad n = \beta T^3
\]

\[
\alpha = \frac{\pi^2 k^4}{15 \hbar^3 c^3} \quad \beta = \frac{2.404 k^3}{\pi^2 \hbar^3 c^3}
\]
The Cosmic Microwave Background

- Review of the CMB
  - almost isotropic (up to 1 part in $10^5$) radiation field
  - blackbody spectrum (the best one we have ever seen), with
    $$T_0 = 2.7255 \pm 0.0006 \text{ K}$$
  - microwave radiation ($\sim 50 - 400$ GHz)
  - redshift $z \sim 1100$

\[ n_\gamma = 4.11 \times 10^8 \text{ m}^{-3} \]
\[ \varepsilon_{\gamma,0} = \alpha T_0^4 = 0.261 \text{ MeV m}^{-3} \]
\[ E_{\text{mean}} = 6.34 \times 10^{-4} \text{ eV} \]
Wavelength [mm]

Intensity [MJy/sr]

FIRAS data with 400σ errorbars

2.725 K Blackbody
The isotropic CMB (2)
Thermodynamic Considerations

- Blackbody CMB → thermal equilibrium
  - We can use equilibrium thermodynamics
  - Consider a co-moving (expanding like the universe) volume $V \propto a(t)^3$.

\[
dQ = dE + PdV = 0 \quad [\text{adiabatic equilibrium}]
\]

\[
\frac{dE}{dt} = -P(t)\frac{dV}{dt}
\]

\[
E = \varepsilon_\gamma V = \alpha T^4 V
\]

\[
P = \frac{\varepsilon_\gamma}{3} = \frac{\alpha T^4}{3}
\]

Exerc!
Thermodynamic Considerations (2)

\[
\alpha \left( 4T^3 \frac{dT}{dt} V + T^4 \frac{dV}{dt} \right) = -\frac{1}{3} \alpha T^4 \frac{dV}{dt}
\]

\[
\frac{1}{T} \frac{dT}{dt} = -\frac{1}{3V} \frac{dV}{dt} = -\frac{d}{dt} \ln V^{1/3}
\]

\[
\frac{d}{dt} \ln T = -\frac{d}{dt} \ln a
\]

\[
T(t) \propto a(t)^{-1}
\]
The isotropic CMB (3)

- Big-bang model in one line: universe starts very hot & dense; it expands; expansion makes it cool & empty.
- The CMB has a redshift $z \sim 1100$
  - $1 + z = a(t_0) / a(t_{em})$
  - $z \sim 1100 \rightarrow \text{universe} \sim 1100 \text{smaller} @ \text{CMB epoch}$
  - $T(t) \propto a(t)^{-1} \rightarrow T \sim 1100 \text{higher} @ \text{CMB epoch}$
  - $T(\text{CMB}) @ \text{emission} \sim 3000 \text{K} \sim 0.26 \text{eV}$
    - $3000 \text{K} \rightarrow \text{near-infrared} \ [T(\text{Sun}) \sim 5800 \text{K}]$
- Simple explanation by the Big-Bang model
  - $0.26 \text{eV} \leftrightarrow \text{Ionization energy hydrogen} (13.6 \text{eV})$
  - Discovery in the 60's: decisive evidence for that model
Exercise!

- If the mean energy $<E>$ of the photons is 0.26 eV, what fraction $f$ of photons have energy $> 13.6$ eV?
  - What must be the mean energy for (i) $f = 0.1$? (ii) $f = 0.001$? (iii) $f = 10^{-(\text{number of letters in your name+lastname})}$?
  - Make a plot of $f$ vs. $<E>$ for a range of $<E>$ encompassing both energies above.
Chapter 3

Newton vs. Einstein

- Equivalence Principle
  - Non-inertial forces

- Curvature in non-euclidean geometries

- The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

- Proper distance
The Equivalence Principle

- Classical physics allows in principle 3 “kinds” of mass
  - Inertial mass \( m_i \)
    \[ F = m_i a \]
  - Active gravitational mass \( m_{g,a} \)
    \[ \Phi = -\frac{GM_{g,a}}{r} \]
  - Passive gravitational mass \( m_{g,p} \)
    \[ F = -\frac{GM_{g,a} m_{g,p}}{r^2} \hat{r}_{mM} \]
- Experiments tell us that the 3 kinds coincide to high precision
The Equivalence Principle (2)

- The weak equivalence principle (WEP) states that $m_i = m_{g,a} = m_{g,p}$.

- Torsion balance experiments provide the most accurate test of the weak equivalence principle.
  - Sensitive to changes in the direction of forces
  - Can test the WEP to 1 part in $10^{13}$
The Equivalence Principle (3)

- Define the Eötvös parameter (see arXiv:1207.2442 - CQG):

\[ \eta_{1,2} = \frac{a_1 - a_2}{(a_1 + a_2)/2} = \frac{(m_g/m_i)_1 - (m_g/m_i)_2}{[(m_g/m_i)_1 + (m_g/m_i)_2]/2} \]

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<tbody>
<tr>
<td>( \Delta a_N )</td>
<td>( 10^{-15} , \text{m s}^{-2} )</td>
<td>0.6 ± 3.1</td>
</tr>
<tr>
<td>( \Delta a_W )</td>
<td>( 10^{-15} , \text{m s}^{-2} )</td>
<td>−2.5 ± 3.5</td>
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<tr>
<td>( \Delta a_\odot )</td>
<td>( 10^{-15} , \text{m s}^{-2} )</td>
<td>−1.8 ± 2.8</td>
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<td>( \Delta a_g )</td>
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<td>−2.1 ± 3.1</td>
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<tr>
<td>( \eta_\oplus )</td>
<td>( 10^{-13} )</td>
<td>0.3 ± 1.8</td>
</tr>
<tr>
<td>( \eta_\odot )</td>
<td>( 10^{-13} )</td>
<td>−3.1 ± 4.7</td>
</tr>
<tr>
<td>( \eta_{DM} )</td>
<td>( 10^{-5} )</td>
<td>−4.2 ± 6.2</td>
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</table>
Define an Yukawa-like correction, with 2 parameters (see arXiv:0712.0607 - PRL):

\[ V(r) = \alpha G \left( \frac{q}{\mu} \right)_1 \left( \frac{q}{\mu} \right)_2 \frac{m_1 m_2}{r_{12}} e^{-r_{12}/\lambda} \]

\[ \lambda = \hbar \left/ (m_b c) \right. \]
So, if the WEP holds exactly, is there a more profound reason?

Gravity force is proportional to the test particle mass

\[ |\mathbf{F}| = -\frac{GM_{g,a}m_{g,p}}{r^2} = -\frac{GMm}{r^2} \propto m \]

What other force(s) has this property?
- Inertial forces!
- Remember classical mechanics, rotating reference frame

\[ \mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_r \]

\[ = \mathbf{F} - m\ddot{\mathbf{r}} - m\dot{\mathbf{\omega}} \times \mathbf{r} - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) - 2m\mathbf{\omega} \times \mathbf{v}_r \]
The Equivalence Principle (5)

- Einstein's idea: maybe gravity is also an inertial force.
  - Generalization of inertial frame → “free-falling”, non-rotating frame
    - No gravity is felt in such a frame
- Strong Equivalence Principle (SEP): The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.
  - Locally: accelerated frame equivalent to non-accelerated frame with gravity
\[ \uparrow a \]

\[ \downarrow g = a \]
The Equivalence Principle (8)

- Gravity must also affect light / photons;
  - Optics: Fermat's Principle → light travels in a path that minimizes travel time
  - With gravity this path is not a straight line
    - Geometry cannot be Euclidean!
    - Space (spacetime) must be **curved**
  - General relativity: free-particles follow **geodesics**
    - 4-D generalization of shortest distance between two points

\[
F - m \sum_{j=1}^{d} \sum_{i=1}^{d} \nu_{j} \Gamma_{ij}^{k} \dot{q}_{i} e_{k} = m \ddot{a},
\]

\[
\frac{d^{2}x^{\alpha}}{d\phi^{2}} + \Gamma^{\alpha}_{\mu\beta} \frac{dx^{\mu}}{d\phi} \frac{dx^{\beta}}{d\phi} = 0
\]
Curvatura

Geometria -> desenvolvida por egípcios, babilônios e chineses.
Gregos deram um tratamento axiomático e definitivo com Euclides (300BC).

Axiomas -> proposições verdadeiras mais simples possível.
Se queremos convencer alguém de que uma afirmação A é correta, devemos mostrar que A segue de B, que esse alguém aceita como verdadeira.
Caso ele não aceite B como verdadeira mostrar que B segue de C e assim por diante, até chegar a uma afirmação que é obviamente verdadeira.

Em sua obra “Os Elementos” Euclides usa 5 postulados ou axiomas e deduz,
junto com certas definições, um total de 465 teoremas.
Um dos postulados de Euclides é o das paralelas.
“Através de um ponto passa uma e somente uma paralela a uma dada linha reta.”
Paralelas -> se em 1 plano 2 linhas não se interceptam então elas são ditas paralelas.
Esse axioma não pode ser simplificado e diferencia o espaço euclidiano de outros espaços.

Geometria euclidiana -> característica: soma dos ângulos internos de 1 triângulo é igual a
$\alpha + \beta + \gamma = 180^\circ$. $C = 2 \pi R$.
Existem inúmeros tipos de espaços. O espaço euclidiano é um dos mais simples.
Ele é dito uniforme. Uniforme = homogêneo e isotrópico. Há invariancia por rotações e translações.
Espaços uniformes -> $C = \gamma \pi e \alpha + \beta + \gamma = j \pi$. Existem + 2 tipos de espaços uniformes.
• Gauss, Lobachevski e Bolyai (geometria hiperbólica)
• Riemann (geometria esférica)
Diferença principal -> existe uma escala de comprimento R característica desse espaço.
Se d<< R o espaço euclidiano é reobtido.
\[ C = 2\pi r < 2\pi l \]

\[ C = 2\pi r > 2\pi l \]
Embora saibamos intuitivamente o que é uma superfície curva, é mais difícil aceitarmos a ideia de curvatura em 3 ou mais dimensões. Isso ocorre pois, p.e., não podemos visualizar um espaço 4-dimensional no qual um 3-espaço apareceria curvo. É possível darmos um significado a curvatura sem referência ao espaço de imersão.

Curvatura intrínseca.
Propriedades intrínsecas → são aquelas que dependem apenas de medidas na superfície. Intrinsecamente uma folha plana é equivalente a 1 cilindro ou um cone. (veja a figura)

Como seres bi-dimensionais descobririam a curvatura de seu mundo? Uma forma é medir comprimento de círculos e áreas.
The Bertrand–Diquet–Puiseux theorem states that for a curve in the plane, the Gaussian curvature $K$ at a point $P$ can be calculated as:

$$K = \frac{3}{\pi} \lim_{r \to 0} \frac{2\pi r}{r^3} \left( -C \right) = \frac{12}{\pi} \lim_{r \to 0} \frac{\pi r^2 - A}{r^4}$$

Where $C$ is a constant and $A$ is the area enclosed by the curve at distance $r$ from $P$. The diagram illustrates the derivation of the formula using polar coordinates:

$$r = a \alpha ; \quad \alpha = \frac{r}{a}$$

$$\theta = \eta \alpha \quad \alpha \quad \theta \quad \text{em radianos}$$

$$\eta = a \theta \operatorname{Sen} \frac{r}{a}$$

$$\operatorname{Sen} x = x - \frac{x^3}{3!} + \ldots$$

$$\eta = \theta \cdot a \left( \frac{r}{a} - \frac{r^3}{6a^3} + \ldots \right) = \theta \left( r - \frac{K}{6} r^3 + \ldots \right)$$

$$C = 2\pi \left( r - \frac{K}{6} r^3 + \ldots \right)$$

$$A = 2\pi \int \left( r - \frac{K}{6} r^3 + \ldots \right) \, dr = \pi \left( r^2 - \frac{Kr^4}{12} + \ldots \right)$$
Line Elements

- The line element $\text{ds}$ tells one how to calculate the distance (interval) between 2 neighboring points.

\[ ds^2 = dx^2 + dy^2 \]
\[ ds^2 = dr^2 + r^2 d\phi^2 \]
These infinitesimal relations can be integrated to yield finite lengths.

- Ex: let's compute the circumference of a circle of radius $R$

$$C = \int dS = \int \left[ (dx)^2 + (dy)^2 \right]^{1/2}$$

$$= 2 \int_{-R}^{+R} dx \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2}_{x^2 + y^2 = R^2}$$

$$= 2 \int_{-R}^{+R} dx \sqrt{\frac{R^2}{R^2 - x^2}}.$$
\[ ds^2 = dx^2 + dy^2 + dz^2 \]
\[ ds^2 = dr^2 + r^2 [d\theta^2 + \sin^2 \theta \, d\phi^2] \]
“Circle” → equidistant points of a given center
  - e.g. lines of constant “latitude”
  - On a sphere (constant positive curvature):

\[ C = \int_0^{2\pi} R \sin \Theta \, d\phi = 2\pi R \sin \Theta \]

\[ r = \int_{\text{center}}^{\text{circle}} \int_0^\Theta Rd\theta = R\Theta \quad C = 2\pi R \sin \left( \frac{r}{R} \right) \]

\[ ds^2 = dr^2 + R^2 \sin^2 \left( \frac{r}{R} \right) d\theta^2 \]

- On a hyperboloid (constant negative curvature):

\[ ds^2 = dr^2 + R^2 \sinh^2 \left( \frac{r}{R} \right) d\theta^2 \]
Line Elements (5)

- In 3D we get
  \[ d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2 \]

  \[ ds^2 = dr^2 + R^2 \sin^2(r/R) d\Omega^2 \]

  \[ ds^2 = dr^2 + r^2 d\Omega^2 \]

  \[ ds^2 = dr^2 + R^2 \sinh^2(r/R) d\Omega^2 \]

- These can be unified as:
  \[ ds^2 = dr^2 + S_\kappa(r)^2 d\Omega^2 \]

  \[ S_\kappa(r) = \begin{cases} 
  R \sin(r/R) & (\kappa = +1) \\
  r & (\kappa = 0) \\
  R \sinh(r/R) & (\kappa = -1) 
\end{cases} \]

- Or, using \( x = S_\kappa(r) \):
  \[ ds^2 = \frac{dx^2}{1 - \kappa x^2/R^2} + x^2 d\Omega^2 \]
The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

- Special relativity tells us how to compute separations in 4-D space-times
  - The separation \( ds^2 = dx^2 + dy^2 + dz^2 \) depends on observer (spatial contraction)!
  - We want to work with invariant (observer-independent) quantities.
  - \( c = \text{const} \rightarrow -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \text{const} \)
  - Special relativity \( \leftrightarrow \) the Minkowski metric
    - Line element: \( ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \)
      - non-euclidean
      - photons: \( ds^2 = 0 \) geodesics (null-geodesics);
      - matter: \( ds^2 < 0 \) geodesics (timelike-geodesics);
The FLRW metric (2)

- We can generalize Minkowski to allow for spatial expansion and/or contraction
  - Assuming isotropy & homogeneity, the most general metric is the FLRW one, with line element:
    \[
    ds^2 = -c^2 dt^2 + a(t)^2 \left[ \frac{dx^2}{1 - \kappa x^2/R_0^2} + x^2 d\Omega^2 \right]
    \]
    \[
    ds^2 = -c^2 dt^2 + a(t)^2 \left[ dr^2 + S_\kappa(r)^2 d\Omega^2 \right]
    \]
- The metric itself is a (rank 2) tensor $g_{\mu\nu}$ with components $g_{\mu\nu}$
  \[
  ds^2 \equiv \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} g_{\mu\nu} dx^\mu dx^\nu
  \]
Proper Distance

- For simplicity, let's consider instantaneous radial separations: \( ds = a(t) \, dr \)
  - The proper distance between two co-moving objects (one at the origin) is:
    \[
    d_p(t) = a(t) \int_0^r dr' = a(t)r
    \]
  - The proper velocity is defined as:
    \[
    v_p(t) \equiv \dot{d}_p(t) = \dot{a}(t)r = \frac{\ddot{a}(t)}{a(t)}d_p(t) \equiv H(t)d_p(t)
    \]
    \[
    v_p(t_0) = H(t_0)d_p(t_0) \equiv H_0d_p(t_0) \quad H(t) \equiv \frac{\ddot{a}(t)}{a(t)}
    \]
Proper Distance (2)

- The proper velocity can be larger than $c$ for faraway galaxies!
  - The critical distance is called the Hubble distance $d_H$

$$v_p(t_0) = c \quad \Rightarrow \quad d_p(t_0) = \frac{c}{H_0} \equiv d_H(t_0)$$

$$d_H(t_0) = (4380 \pm 130) \text{ Mpc}$$

- The proper distance is **unobservable** (not in the past lightcone)
  - we can only observe it indirectly (assuming a model)

- However, the scale factor at emission $a(t_{em})$ is observable
**Proper Distance (3)**

- Let's see what happens to a photon over large distances
  - Without loss of generality, let's consider radial null-geodesics separations: $ds = 0 \rightarrow a(t) \, dr = \pm c \, dt$
  - Define: emission time ($t_e$) and reception time ($t_0$)

\[
\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r
\]

\[
\int_{t_e + \lambda_e/c}^{t_0} \frac{dt}{a(t)} = \int_{t_e + \lambda_e/c}^{t_0 + \lambda_0/c} \frac{dt}{a(t)}
\]

If we subtract this:

\[
\int_{t_e + \lambda_e/c}^{t_0} \frac{dt}{a(t)}
\]

We get this:

\[
\int_{t_e}^{t_0 + \lambda_0/c} \frac{dt}{a(t)} = \int_{t_0}^{t_0 + \lambda_0/c} \frac{dt}{a(t)}
\]
Proper Distance (4)

The integrand is effectively constant \((\lambda/c \sim 10^{-14} \text{ s} \sim 10^{-32} \text{ } H_0^{-1})\)

\[
\int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\lambda_0/c} \frac{dt}{a(t)}
\]

\[
\frac{1}{a(t_e)} \int_{t_e}^{t_e+\lambda_e/c} dt = \frac{1}{a(t_0)} \int_{t_0}^{t_0+\lambda_0/c} dt
\]

\[
\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)}
\]

\[
z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}
\]

\[
1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)}
\]
Chapter 4
Cosmic Dynamics

- Einstein's equations
  - The Friedmann equation
- Fluid equations
- Equations of state
- The cosmological constant
Measuring the curvature

- In a homogenous & isotropic universe, all information is contained in: \( a(t) \), \( \kappa \) and \( R_0 \) (if \( \kappa \neq 0 \)).
  - \( \kappa = +1 \) → spherical (Riemann) geometry
  - \( \kappa = 0 \) → flat geometry
  - \( \kappa = -1 \) → hyperbolical (Lobachevski) geometry
- The radius of curvature \( R_0 \) must be comparable with the Hubble radius \( d_H \)
  - Otherwise we would see multiple images of galaxies
Einstein's Equations

- The fundamental equations of general relativity are Einstein's equations (10 second order partial diff. eqs.)
  - Tensors are algebraic constructs in vector spaces, independent of coordinate systems (observers)
  - The **metric** is a function of the energy-momentum tensor \( T \)

\[
G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} \quad (\mu, \nu = \{0, 1, 2, 3\})
\]

\[
G_{\mu\nu} \left( g, \partial g, \partial \partial g \right) = G_{\nu\mu} \quad \text{(symmetric)}
\]

\[
\sum_{\mu} \nabla^\mu G_{\mu\nu} = 0 = \sum_{\mu} \nabla^\mu T_{\mu\nu} \quad \text{(energy-momentum conservation)}
\]
Einstein's Equations (2)

- It is often convenient to work on abstract-index notation
  - Tensors are represented by their coordinates in an undefined coordinate system
    - Sometimes implicitly assumed to be cartesian
    - Can be written in either covariant (lower indices) or contravariant (upper indices) form

\[
G = \frac{8\pi G}{c^2} \, T \quad \rightarrow \quad G_{\mu\nu} = \frac{8\pi G}{c^2} \, T_{\mu\nu}
\]

- In a homogeneous and isotropic universe (FLRW metric), only 2 of the above equations are non-zero & independent

\[
G_{00} = \frac{8\pi G}{c^2} \, T_{00} \quad G_{11} = G_{22} = G_{33} = \frac{8\pi G}{c^2} \, T_{11}
\]
Einstein's Equations (3)

- In GR, we can **raise** and **lower** indexes by internal product with the metric $g$

\[ \sum_{\alpha} g^{\alpha \mu} g_{\alpha \nu} \equiv \delta^\mu_\nu \quad G^{\mu \nu} \equiv \sum_{\alpha} g^{\alpha \mu} G_{\alpha \nu} = \frac{8 \pi G}{c^2} T^{\mu \nu} \]

- In cosmology it is more **convenient** to work with mixed raised/lowered indices (one up, one down)

- In FLRW spacetimes, matter must behave like a perfect fluid

\[ T^{\mu \nu} = (\varepsilon + P) u^\mu u_\nu + P \delta^\mu_\nu \]

- In comoving cartesian coordinates we get:

\[ u^\mu = (-1, 0, 0, 0) \quad g_{\mu \nu} = \text{diag}(-1, a^2, a^2, a^2) \]

\[ u_\mu = (1, 0, 0, 0) \quad T^{\mu \nu} = \text{diag}(-\varepsilon, P, P, P) \]
Friedmann Equation

- For a Newtonian derivation, see [Ryden]
- From Einstein's 00 (time-time) equation we have the so-called Friedmann equation

\[ G^0_0 = \frac{8\pi G}{c^2} T^0_0 \]

\[ -3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2} \right] = -\frac{8\pi G}{c^2} \varepsilon(t) \]

\[ H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2} \]
Friedmann Equation (2)

\[ H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2} \]

- Since energy is positive, for \( \kappa = -1 \), there is a minimum radius of curvature for the universe

\[ R_0(\text{min}) = \frac{c}{H_0} = d_H(t_0) \]

- For spatially flat universes, there is a critical density related to the Hubble parameter

\[ \varepsilon_{\text{crit}}(t) = \frac{3c^2}{8\pi G} H(t)^2 \quad \varepsilon_{\text{crit},0} = (4870 \pm 290) \text{ MeV m}^{-3} \]
Friedmann Equation (3)

- The universe has a **very low** average density
  - The critical density is roughly:
    - 1 hydrogen / 200 liters
    - 140 solar masses / kpc$^3$

\[
\rho_{\text{crit},0} \equiv \frac{\varepsilon_{\text{crit},0}}{c^2} = (8.7 \pm 0.5) \times 10^{-27} \text{ kg m}^{-3}
\]

\[
= (1.28 \pm 0.08) \times 10^{11} \text{ M}_\odot \text{ Mpc}^{-3}
\]
Friedmann Equation (4)

- In the Einstein Equation, the energy-momentum tensor $T$ is the total energy-momentum tensor, the sum of different $T$'s for all species (photons, baryons, neutrinos, dark matter, etc.)

$$H(t)^2 = \frac{8\pi G}{3c^2} \sum_i \varepsilon_i(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}$$

- It is convenient to define normalized density parameters

$$\Omega_i(t) \equiv \frac{\varepsilon_i(t)}{\varepsilon_{\text{crit}}(t)} \quad \Omega_{\text{tot}}(t) \equiv \Omega(t) \equiv \sum \Omega_i(t)$$

$$\varepsilon_{\text{tot}}(t) \equiv \varepsilon(t) \equiv \sum \varepsilon_i(t)$$
The Friedmann equation is thus rewritten as

$$\sum_i \Omega_i(t) = 1 + \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2 H(t)^2}$$

We can also treat curvature as an effective fluid, with associated energy density and pressure

$$\Omega_\kappa(t) = -\frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2 H(t)^2} \sum_i \Omega_i(t) = 1 - \Omega_\kappa(t)$$

In particular, at present we have

$$\sum_i \Omega_{i0} = 1 - \Omega_{\kappa 0} = 1 + \frac{\kappa c^2}{R_0^2 H_0^2}$$
Fluid Equations

- From Einstein's $ij$ (space-space) equations we have the so-called \textit{acceleration equation}.

- For a Newtonian derivation, again see [Ryden]

\[ G^i_j = \left( \frac{8\pi G}{c^2} \right) T^i_j \]

\[- \left( H^2 + 2 \frac{\ddot{a}}{a} - \frac{\kappa c^2}{R_0^2 a^2} \right) \delta^i_j = \frac{8\pi G}{c^2} P \delta^i_j = \frac{8\pi G}{3c^2} 3P \delta^i_j \]

Friedmann Eq. $\rightarrow$

\[ H^2 - \frac{\kappa c^2}{R_0^2} \frac{1}{a^2} = \frac{8\pi G}{3c^2} \varepsilon \]

\[ \frac{\ddot{a}}{a} = - \frac{4\pi G}{3c^2} \left[ \varepsilon + 3P \right] \quad \text{“acceleration equation”} \]
We have derived 2 fundamental equations so far:

\[ H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2} \]

“Friedmann equation”

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \left[ \varepsilon + 3P \right] \]

“acceleration equation”

We can combine them to derive the conservation equation (or simply fluid equation)

\[ \dot{\varepsilon} + 3\frac{\dot{a}}{a} \left[ \varepsilon + P \right] = 0 \]

Problem: we have 2 independent eqs. and 3 variables

- We need another equation!
Equations of State

- Normal fluids typically have a well-defined equation of state (EoS), which relates its pressure with other thermodynamic quantities, like its energy $E$ or entropy $S$

$$P = P(\varepsilon, S)$$

- The equation of state can be non-linear and very complicated in general (like in condensed matter or in stars)
- In cosmology, we deal with dilute gases, with very simple EoS

$$P_i = w_i \varepsilon_i \quad (w_i \rightarrow \text{constant})$$

- We will allow for $w(t)$ in the future, but we will neglect any dependence of $P$ in entropy throughout this course
Equations of State (2)

- In cosmology → typically very simple EoS
  \[ P_i = w_i \varepsilon_i \quad (w_i \rightarrow \text{constant}) \]

- Low density gas → ideal gas
  - Consider an ideal & non-relativistic gas (& particle-mass \( \mu \))
    \[ P = \frac{N}{V} k_B T = \frac{\rho}{\mu} k_B T \]
    \[ \varepsilon \approx \rho c^2 \]

- Maxwell's velocity distribution → \[ 3k_B T = \mu \langle v^2 \rangle \]
  \[ P_{\text{nonrel}} = \frac{\langle v^2 \rangle}{3c^2} \varepsilon \ll \varepsilon \quad [w_{\text{nonrel}} \approx 0] \]
Equations of State (3)

- Fully relativistic matter has instead:
  \[ P_{\text{rel}} = \frac{1}{3} \varepsilon_{\text{rel}} \]
  \[ w_{\text{rel}} = \frac{1}{3} \]

- Mildly relativistic matter is in-between \(0 < w < 1/3\)

- Curvature has effectively: \(w_\kappa = -1/3\)

- As we will see, a Cosmological Constant \(\Lambda\) is described by
  \[ P_\Lambda = -\varepsilon_\Lambda \]
  \[ w_\Lambda = -1 \]
Equations of State (4)

- A perturbation in the fluid generates sound waves
  - For adiabatic perturbations, the speed of sound is
    \[ c_s^2 = c^2 \left( \frac{dP}{d\varepsilon} \right) \rightarrow c_s = \sqrt{\omega c} \]

- Causality \( \rightarrow w \leq 1 \)
- \( w < 0 \rightarrow \) exponential perturbations

- In general, dark energy refers to any fluid with \( w < -\frac{1}{3} \)
  - Observations \( \rightarrow w_{\text{DE}} \approx -1.0 \pm 0.1 \)
The Cosmological Constant $\Lambda$

- Historically, $\Lambda$ was introduced by Einstein in 1917 to produce a static universe
  - By 1917 it was still unclear whether there existed other galaxies beyond the Milky Way
    - What was the distance to the observed nebulae?
    - By mid-1920's, Ernst Öpik and Edwin Hubble determined that Andromeda was far outside our galaxy

- Einstein modified his equations by adding a constant term

$$G^\mu_\nu - \Lambda \delta^\mu_\nu = \frac{8\pi G}{c^2} T^\mu_\nu$$
The Cosmological Constant \( \Lambda \) (2)

- The modified fluid equations are:

\[
H^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0} \frac{1}{a^2} + \frac{\Lambda}{3}
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \left[ \varepsilon + 3P \right] + \frac{\Lambda}{3}
\]

\[
\dot{\varepsilon} + 3\frac{\dot{a}}{a} \left[ \varepsilon + P \right] = 0
\]

- The \( \Lambda \) terms can be absorbed into \( \varepsilon \) & \( P \) by identifying

\[
\varepsilon_\Lambda \equiv \frac{c^2}{8\pi G} \Lambda \quad P_\Lambda \equiv -\varepsilon_\Lambda = -\frac{c^2}{8\pi G} \Lambda
\]
The Cosmological Constant $\Lambda$ (3)

- A static universe requires $\dot{a} = \ddot{a} = 0$
- Einstein assumed (correctly) that in the present universe matter ($P = 0$) was dominant over radiation or curvature
  - Energy in starlight << rest-energy of stars
    \[
    \frac{\ddot{a}}{a} = 0 = -\frac{4\pi G}{3c^2} [\varepsilon + 0] + \frac{\Lambda}{3} \quad \rightarrow \quad \Lambda = 4\pi G \rho
    \]
    \[
    H^2 = 0 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0^2} \frac{1}{a^2} + \frac{\Lambda}{3} \quad \rightarrow \quad R_0 = \frac{c}{\Lambda^{1/2}}
    \]
  - Einstein's static universe had to be closed (Riemannian)!
The Cosmological Constant $\Lambda$ (4)

- By late 1920's, the universe was observed to be expanding.
- Einstein's static universe had a bigger flaw → instability!
  - In the newtonian limit we have a modified Poisson eq.:
    \[ \nabla^2 \Phi + \Lambda = 4\pi G \rho \n\]
  - Repulsive force from $\Lambda$ balances attractive gravity from $\varrho$.
  - Small perturbations from $\Lambda = 4\pi G \rho$ lead to expansion or collapse.
  - Einstein: "the greatest blunder of my career".
- Although a static universe is discarded, $\Lambda$ had a good side effect: it could make the Universe older.
The Cosmological Constant $\Lambda$ (5)

- Although a static universe is discarded, $\Lambda$ had a good side effect: it could make the Universe older.
  - Lemaître & Hubble measured $H_0 \sim 500 \frac{\text{km}}{\text{s Mpc}}$.
  - The simple estimate for the age of the universe becomes too small (smaller than the age of the Earth): $H_0^{-1} \sim 2 \text{ Gyr}$.

- A positive $\Lambda$ can make the universe accelerate at late times, and thus $H(t)$ was smaller in the past (recall slide #16).

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \left[ \varepsilon + 3P \right] + \frac{\Lambda}{3} > 0
\]
The Cosmological Constant $\Lambda$  (6)

- Late 1990's: $\Lambda$ reappears due to late accelerated expansion
- What could be behind $\Lambda$? Are there good physical candidates with $w = -1$?
  - **Answer:** YES! → vacuum (rest point) energy!
  - Vacuum energy should **not** depend on the expansion of the universe → constant $\varepsilon$
- Let's compute it from quantum field theory
  \[
  \langle \varepsilon_{\text{vac}} \rangle = \int_{0}^{\Lambda \gg m} \frac{dk}{(2\pi)^3} \frac{4\pi k^2}{2} \frac{1}{\sqrt{k^2 + m^2}} \sim \frac{\Lambda^4}{16\pi^2}
  \]
- A conservative cutoff at the LHC scale → $\langle \varepsilon_{\text{vac}} \rangle \sim 10^{56} \varepsilon_{\Lambda}$
  - See e.g. S. Weinberg's book *Cosmology*, p. 56
- Note that this is a very naïve calculation: see [1205.3365](https://arxiv.org/abs/1205.3365)
- End of Part I -